A Closed-Form Approximation to the Arithmetic-Geometric Mean

The Arithmetic-Geometric Mean (AGM) is an average defined by the substitutionary series[1]:-

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}$$

Equation 1

for the positive arguments a_0, b_0.
Because:-

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Equation 2

for all positive a, b the nested iteration converges to a limit:-

$$a_\infty = b_\infty = AGM(a_0, b_0)$$

Equation 3

In practical terms, however, the series converges to an estimate accurate to fifteen figures after no more than seven iterations and usually far fewer: The process is quadratically convergent.

Because of the nested substitutionary character of this estimative algorithm it is possible to represent it as a fractal dendroid of nesting convergent formulae.

AGM_0 is the zeroth iterate of this developing tree structure, where $x \equiv a_0$ and $y \equiv b_0$:-

$$AGM_0 = \frac{\frac{x+y}{2} + \sqrt{xy}}{2}$$

Equation 4

By substituting:-

$$\frac{x+y}{2} \quad for \quad x$$
$$\sqrt{xy} \quad for \quad y$$

we can develop this structure into AGM_1, which is illustrated below:-

$$AGM_1 = \frac{\frac{\frac{x+y}{2}+\sqrt{xy}}{2} + \sqrt{\frac{x+y}{2}\times\sqrt{xy}}}{2}$$

Equation 5

Iterative dendroids such as these may be expressed as algebraic series and that for AGM_1 is given here:-

$$AGM_1 = \frac{1}{8}x + \frac{1}{8}y + \frac{1}{4}.\sqrt{x}\sqrt{y} + \frac{1}{4}.\sqrt{2}\sqrt{x+y}.x^{\frac{1}{4}}.y^{\frac{1}{4}}$$

Equation 6

To choose a phytological metaphor, the solution buds at the growing meristem of the fractal tree, leaving behind dead branches; yet thus being more animate and coralline than vegetative and arboreal.

The AGM blossoms forth on the ends of the fractal dendroids' branches in ever-converging efflorescences of approximation, and this progressive refinement is illustrated in an interesting property of the series expression: Twice the final term is very nearly the AGM.

If AGM_i is the i^{th} iterative approximation of the AGM and S_i is the Series Summation of iteration i then we may write:-

$$\begin{aligned} S_i &= t_1 + t_2 + t_3 + \ldots + t_{i+3} \\ &= t_1 + t_2 + t_3 + \ldots + t_{i+2} + U_i \\ &= \sum_{j=1}^{i+2} t_j + U_i \end{aligned}$$

Equation 7

and:-

$$AGM_\infty \approx AGM_i \approx 2U_i$$

Equation 8

The latter additive series terms, and especially U_i, rapidly grow to enormous lengths as iteration i is increased, but as we shall see substitutions and simplifications enable the Ultimate Term U_i to be formed into a manageable equivalent.

Nevertheless, the simplicity and rapid convergence of the classical defining algorithm render this closed form estimate $2U_i$ unappealing as a computational resource.

It virtue, if any, shall inhere in its convenience as a component of other approximations, perhaps in the estimation of elliptic integrals.

Borwein and Zucker[2] have extended the AGM analytic procedures formulated to define Gamma Function values for rational fractional arguments. Closed form approximations of the AGM or other iterative functions may have a rôle in the future estimation of Gamma Functions for very small arguments. We may now view the dendroid and series expressions for the second-order iteration at i = 2 which are respectively:-

$$AGM_2 = \frac{\dfrac{\dfrac{\dfrac{x+y}{2}+\sqrt{xy}}{2}+\sqrt{\dfrac{x+y}{2}\times\sqrt{xy}}}{2}+\sqrt{\dfrac{\dfrac{x+y}{2}+\sqrt{xy}}{2}\times\sqrt{\dfrac{x+y}{2}+\sqrt{xy}}}}{2}$$

Equation 9

and:-

$$AGM_2 = \frac{1}{16}x + \frac{1}{16}y + \frac{1}{8}.\sqrt{x}\sqrt{y} + \frac{1}{8}.\sqrt{2}\sqrt{x+y}.x^{\frac{1}{4}}.y^{\frac{1}{4}} + \frac{1}{8}.2^{\frac{3}{4}}\sqrt{x+y+2\sqrt{x}\sqrt{y}}.(x+y)^{\frac{1}{4}}.x^{\frac{1}{8}}.y^{\frac{1}{8}}$$

Equation 10

It is impracticable to present any higher-order developments in their totality within a normal document:- Due to their massive lengths.

We may note, however, the following general characteristics of the Ultimate Terms U_i:-

(a) A Leading Power Group, P_a:-

$$P_a = \frac{1}{2^k}.2^{\frac{l}{m}}$$

Equation 11

(b) A Middle Nested Root Group, R_a:-

$$R_a = \sqrt{x+y+2\sqrt{xy}}$$

Equation 12

or a development of such a structure.

(c) A Trailing Argument Power Group, A_a:-

$$A_a = (x+y)^{\frac{1}{m}}x^{\frac{2}{m}}y^{\frac{2}{m}}$$

Equation 13

or a development of such a structure.

k, l and m are related integers which may be computed from i, the Iteration Order.

Numerical Fiducials and Sufficient Approximation

A MATHCAD experiment was performed with $x = a_0$ set to 12011.4999 and $y = b_0$ set to 97.112004998.

These arbitrary but widely-separated figures were chosen to identify an order of approximation adequate to furnish several figures of accuracy with the incorporation of a secondary approximation $\mathrm{Sin}(\theta) \approx \theta$.

A MATHCAD iteration of classical type was defined for i = 0...7 and took the form:-

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 12011.4999 \\ 97.112004998 \end{pmatrix}$$

Equation 14

$$\begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} = \begin{pmatrix} \dfrac{a_i + b_i}{2} \\ \sqrt{a_i b_i} \end{pmatrix}$$

Equation 15

This converged to at least twelve decimal places of accuracy after six iterations as pictured in the tabulation below:-

i	a_i	b_i
0	12011.4999	97.112004998
1	6054.305952499	1080.028165522676
2	3567.167059010838	2557.111837873055
3	3062.139448441946	3020.206137711038
4	3041.172793076492	3041.100517363978
5	3041.136655220235	3041.136655005522
6	3041.136655112879	3041.136655112879
7	3041.136655112879	3041.136655112879

Table One
An AGM Iteration Tabulation

Accordingly, a_7 was selected as the fiducial estimate of the given values of x and y:-

$$a_7 = 3041.136655112879$$

Equation 16

For convenience we will define the Ultimate Term Estimate, $V_i \equiv 2 \times U_i$.

The Relative Error, $\rho(V_i)$, will be defined as:-

$$\rho(V_i) = \frac{V_i - a_7}{V_i}$$

Equation 17

Further experiments permitted the following error tabulation to be developed:-

i	V_i	$\rho(V_i)$
0	1080.028165522676	-1.815793839636702
1	2257.111837873055	-0.189285744202109
2	3020.206137711038	-0.006930161865608
3	3041.100517363979	-0.000011883115567
4	3041.136655005522	-0.000000000035301
5	3041.136655112879	0

Table Two
Relative Error of the Ultimate Term Estimate

These experiments confirm that for these extreme arguments, V_5 is an adequate estimate of AGM_∞ to twelve-figure accuracy.

According U_i is presented piecemeal as the product of:-

(a) $P_a \equiv$ AGM5T8A

$$\mathrm{AGM5T8A} := \frac{1}{64} \cdot 2^{\frac{31}{32}}$$

Equation 18

(b) $R_a \equiv$ AGM5T8B

$$\mathrm{AGM5T8B} := \sqrt{x+y+2\cdot\sqrt{x}\cdot\sqrt{y}+2\cdot\sqrt{2}\cdot\sqrt{x+y}\cdot x^{\frac{1}{4}}\cdot y^{\frac{1}{4}}+2\cdot 2^{\frac{3}{4}}\cdot\sqrt{x+y+2\cdot\sqrt{x}\cdot\sqrt{y}}\cdot(x+y)^{\frac{1}{4}}\cdot x^{\frac{1}{8}}\cdot y^{\frac{1}{8}}+2\cdot 2^{\frac{7}{8}}\cdot\sqrt{x+y+2\cdot\sqrt{x}\cdot\sqrt{y}+2\cdot\sqrt{2}\cdot\sqrt{x+y}\cdot x^{\frac{1}{4}}\cdot y^{\frac{1}{4}}}\cdot\left(x+y+2\cdot\sqrt{x}\cdot\sqrt{y}\right)^{\frac{1}{4}}\cdot(x+y)^{\frac{1}{8}}\cdot x^{\frac{1}{16}}\cdot y^{\frac{1}{16}}}$$

Equation 19

(c) $A_a \equiv$ AGM5T8C

$$\mathrm{AGM5T8C} := \left[x+y+2\cdot\sqrt{x}\cdot\sqrt{y}+2\cdot\sqrt{2}\cdot\sqrt{x+y}\cdot x^{\frac{1}{4}}\cdot y^{\frac{1}{4}}+2\cdot 2^{\frac{3}{4}}\cdot\sqrt{x+y+2\cdot\sqrt{x}\cdot\sqrt{y}}\cdot(x+y)^{\frac{1}{4}}\cdot x^{\frac{1}{8}}\cdot y^{\frac{1}{8}}\right]^{\frac{1}{4}}\cdot\left(x+y+2\cdot\sqrt{x}\cdot\sqrt{y}+2\cdot\sqrt{2}\cdot\sqrt{x+y}\cdot x^{\frac{1}{4}}\cdot y^{\frac{1}{4}}\right)^{\frac{1}{8}}\cdot\left(x+y+2\cdot\sqrt{x}\cdot\sqrt{y}\right)^{\frac{1}{16}}\cdot(x+y)^{\frac{1}{32}}\cdot x^{\frac{1}{64}}\cdot y^{\frac{1}{64}}$$

Equation 20

The actual estimate V_i is accordingly given by:-

$$V_i = 2U_i = 2 \times AGM5T8A \times AGM5T8B \times AGM5T8C$$

Equation 21

The Simplification of V_5

It is possible to simplify the ultimate terms U_i to achieve employable estimators in at least these three idioms:-

(a) Algebraic
(b) Logarithmic
(c) Trigonometric

No pretence is made for the computational economy of any assembly in this AGM context.

Because of its likely fecundity, especially in the facilitation of accurate secondary approximations, we will pursue the trigonometric route and develop V_5 simplification in three successive phases.

Trigonometric Resolution Phase One

This phase requires the definition of the auxiliary substitutive functions:-

$$v = \sqrt{(x+y)^2 - (x-y)^2} = \sqrt{4xy} = 2\sqrt{xy}$$

Equation 22

$$\theta = Cos^{-1}\left(\frac{x-y}{x+y}\right)$$

Equation 23

$$v = Sin\theta(x+y)$$

Equation 24

Now from these three it follows that:-

$$x+y = \frac{v}{Sin\theta}$$

Equation 25

$$\sqrt{xy} = \frac{v}{2}$$

Equation 26

and an appropriate substitution gives:-

$$V_5 = AP5 = f(Sin\theta) \quad only$$

Equation 27

AP5 is however very long and it is not helpful to explicate it here.

Trigonometric Resolution Phase Two

Because AP5 is a function of Sinθ (and powers of two) only, several of its internal terms can be co-ordinated using these definitions and identities:-

$$\psi = \sqrt{Sin\theta}$$

Equation 28

$$\chi = 1 + \psi$$

Equation 29

$$\omega = 1 + \psi^2$$

Equation 30

$$\chi^2 = (1+\psi)^2 = 1 + 2\psi + \psi^2$$

Equation 31

$$\sqrt{1+\psi^2} \times \sqrt{\psi} = \sqrt{\psi(1+\psi^2)} = \sqrt{\psi\omega}$$

Equation 32

$$(1+\psi^2)^{\frac{1}{4}}.\omega^{\frac{1}{4}} = \sqrt[4]{\psi(1+\psi^2)}$$

Equation 33

$$(1+\psi^2)^{\frac{1}{4}}.\psi^{\frac{5}{4}} = (1+\psi^2)^{\frac{1}{4}}.\psi^{\frac{1}{4}}.\psi = \sqrt[4]{\psi(1+\psi^2)} \times \psi$$

Equation 34

Substitution yields an assembly co-ordinated in terms of $(1+\psi)^2$, $\sqrt{\psi(1+\psi^2)}$ and $\sqrt[4]{\psi(1+\psi^2)}$. This expression (which is almost short enough to fit across an A4 page) we will call AP8.

Replacement of $(1+\psi)^2$ with χ^2 and $\psi(1+\psi^2)$ with $\psi\omega$ enables us now to show AP8 as:-

$$AP8 = \frac{1}{32}.2^{\frac{15}{16}}.v.\frac{\sqrt{\chi + 2^{\frac{3}{2}}\sqrt{\psi\omega} + 2^{\frac{7}{4}}\sqrt[4]{\psi\omega} + 2^{\frac{7}{4}}\sqrt[4]{\psi\omega}.\omega}}{\psi^{\frac{31}{16}}}.\left(\chi + 2^{\frac{3}{2}}\sqrt{\psi\omega}\right)^{\frac{1}{4}}.\chi^{\frac{1}{4}}.\omega^{\frac{1}{16}}$$

Equation 35

Trigonometric Resolution Phase Three

Further simplification of the V_5 estimator beyond AP8 involves the factoring of the quadratic equation in $\sqrt{\psi\omega}$,which is under the major square root sign.

With minor re-arrangement we may write this quadratic as:-

$$q_0 = 2^{\frac{3}{2}}\sqrt{\psi\omega} + 2^{\frac{7}{4}}(1-\psi)\sqrt[4]{\psi\omega} + \chi^2$$

Equation 36

If we define the auxiliary functions:-

$$f(\psi) = \sqrt{\psi\omega}$$

Equation 37

$$g(\psi) = 1 + \psi$$

Equation 38

we may conveniently re-draft the quadratic as:-

$$q_1 = 2^{\frac{3}{2}}.f(\psi) + 2^{\frac{7}{4}}.g(\psi).\sqrt{f(\psi)} + g(\psi)^2$$

Equation 39

which may now be solved in g(ψ).
The roots of Equation 39 are given by:-

$$r_1[q_1], r_2[q_2] = \frac{-2^{\frac{7}{4}}\sqrt{f(\psi)} + \sqrt{2^{\frac{14}{4}}.f(\psi) - 2^{\frac{7}{2}}.f(\psi)}}{2}$$

Equation 40

and because the $\sqrt{b^2 - 2ac}$ term is zero both roots are given by:-

$$r_1[q_1], r_2[q_2] = -2^{\frac{3}{4}}\sqrt{f(\psi)}$$

Equation 41

Therefore:-

$$\left(g(\psi) + 2^{\frac{3}{4}}\sqrt{f(\psi)}\right)^2 = 2^{\frac{3}{2}}f(\psi) + 2^{\frac{7}{4}}.g(\psi).\sqrt{f(\psi)} + g(\psi)^2$$

Equation 42

Enabling us to specify the much-shortened V_5 estimator:-

$$AP11 = \frac{1}{32}.2^{\frac{15}{16}}.v.\frac{\left[\chi + 2^{\frac{3}{4}}(\psi\omega)^{\frac{1}{4}}\right]}{\psi^{\frac{31}{16}}}\left(\chi^2 + 2^{\frac{3}{2}}\sqrt{\psi\omega}\right)^{\frac{1}{4}}.\chi^{\frac{1}{4}}.\omega^{\frac{1}{16}}$$

Equation 43

Substitution for v and subsequent tidying of the powers of two allows us to set down V_5 as:-

$$AP12 = 2^{-4}2^{\frac{15}{16}}.\sqrt{xy}.\frac{\left[\chi \;\; + 2^{\frac{3}{4}}(\psi\omega)^{\frac{1}{4}}\right]}{\psi^{\frac{31}{16}}}\left(\chi^2 + 2^{\frac{3}{2}}\sqrt{\psi\omega}\right)^{\frac{1}{4}}.\chi^{\frac{1}{4}}.\omega^{\frac{1}{16}}$$

Equation 44

An Estimator of the AGM for Widely-Separated Arguments

When x » y we can approximate the AGM by entering the secondary approximation θ ≈ Sinθ into Equation 44.

This gives:-

$$AP12^* = 2^{-4}2^{\frac{15}{16}}.\sqrt{xy}.\frac{\left[\left(1+\sqrt{\theta}\right) + 2^{\frac{3}{4}}\left(\sqrt{\theta}(1+\theta)\right)^{\frac{1}{4}}\right]}{\sqrt{\theta}^{\frac{31}{16}}}\left(\left(1+\sqrt{\theta}\right)^2 + 2^{\frac{3}{2}}\sqrt{\sqrt{\theta}(1+\theta)}\right)^{\frac{1}{4}}.\left(1+\sqrt{\theta}\right)^{\frac{1}{4}}.(1+\theta)^{\frac{1}{16}}$$

Equation 45

Further simplifications may of course be effected.

Notation

a_i	An Upper Iterate Value
a_7	The Fiducial Estimate of AGM_∞
AGM	The Arithmetic-Geometric Mean
b_i	A Lower Iterate Value
θ	Iteration Bounds' Argument Function
i	The Iteration Number
q_0	Denominator Quadratic in $\sqrt{\psi\omega}$
q_1	Denominator Quadratic in $g(\psi)$
$\rho(V_i)$	The Relative Error of Estimation (of the AGM)
$r_{1,2}[q]$	The First, Second Root of Quadratic q
S_i	The Elaboration Series Sum at Iteration i
t_j	The j^{th} Elaboration Series Term
U_i	The Ultimate Series Term
v	Iteration Bounds Modulus Function
V_i	The Ultimate Series Term Estimate of the AGM
x	The Upper Bounding AGM Argument
y	The Lower Bounding AGM Argument

Further, auxiliary functions are defined in the text.

References

1 “CRC Concise Encyclopedia of Mathematics”
Eric W Weisstein
CRC Press of London 15 October 1998
ISBN 0849396409

http://mathworld.wolfram.com/SquareRoot.html

2 “Elliptic Integral Evaluation of the Gamma Function at Rational Values of Small Denominator”
JM Borwein and IJ Zucker
IMA Journal of Numerical Analysis 1992
Volume 12 Pages 519-526

APPENDIX A

The Fifth Iterate Ultimate Term, U_5, in Full

$$U_5 = \frac{1}{64}\cdot 2^{\frac{31}{32}}$$

$$\sqrt{x+y+2\cdot\sqrt{x}\cdot\sqrt{y}+2\cdot\sqrt{2\cdot\sqrt{x+y}\cdot x^{\frac{1}{4}}\cdot y^{\frac{1}{4}}}+2\cdot 2^{\frac{3}{4}}\cdot\sqrt{x+y+2\cdot\sqrt{x}\cdot\sqrt{y}}\cdot(x+y)^{\frac{1}{4}}\cdot x^{\frac{1}{8}}\cdot y^{\frac{1}{8}}+2\cdot 2^{\frac{7}{8}}\cdot\sqrt{x+y+2\cdot\sqrt{x}\cdot\sqrt{y}+2\cdot\sqrt{2\cdot\sqrt{x+y}\cdot x^{\frac{1}{4}}\cdot y^{\frac{1}{4}}}}\cdot\left(x+y+2\cdot\sqrt{x}\cdot\sqrt{y}\right)^{\frac{1}{4}}\cdot(x+y)^{\frac{1}{8}}\cdot x^{\frac{1}{16}}\cdot y^{\frac{1}{16}}}$$

$$\left[x+y+2\cdot\sqrt{x}\cdot\sqrt{y}+2\cdot\sqrt{2\cdot\sqrt{x+y}\cdot x^{\frac{1}{4}}\cdot y^{\frac{1}{4}}}+2\cdot 2^{\frac{3}{4}}\cdot\sqrt{x+y+2\cdot\sqrt{x}\cdot\sqrt{y}}\cdot(x+y)^{\frac{1}{4}}\cdot x^{\frac{1}{8}}\cdot y^{\frac{1}{8}}\right]^{\frac{1}{4}}\cdot\left(x+y+2\cdot\sqrt{x}\cdot\sqrt{y}+2\cdot\sqrt{2\cdot\sqrt{x+y}\cdot x^{\frac{1}{4}}\cdot y^{\frac{1}{4}}}\right)^{\frac{1}{8}}\cdot\left(x+y+2\cdot\sqrt{x}\cdot\sqrt{y}\right)^{\frac{1}{16}}\cdot(x+y)^{\frac{1}{32}}\cdot x^{\frac{1}{64}}\cdot y^{\frac{1}{64}}$$