Pseudoanalytic Mensuration of The Castrum By Means of The Ridders Method

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PART ONE DEFINITION

There is a shape on the Euclidean plane that is like a rectangle with quadrant corners. I do not know the conventional name of this shape. Some people call it a playing-card shape, and some a filleted rectangle. I shall call it a castrum, the Roman word for an encampment, because the figure suggests the usual plan of such a structure.

Accurate mensuration of such a shape is problematical, because whilst theoretically fixed by knowledge of three points, quadrant radius is difficult to measure.

Therefore, an indirect but accurate method for the determination of fillet radius is desirable.

The method I present depends upon three linear data only: x, the Width of the castrum measured normal to its vertical straight sides; y, the Height of the castrum measured normal to its horizontal straight sides; and d, the "Calipered" Diagonal, a cross lineament parallel to the bisectors of opposite quadrants, between the extreme tangents of those quadrants.

Figure One, in which the blue and green circuit delineates the castrum, illustrates definitional lines and angles. In the context of this diagram, and the other figure diagrams of Appendix A it should be noted that whilst the radius value is definitional, the d, f and h values noted in blue are computed from measured paper and are of only three-figure significance.

The areal zones i, ii, iii, iv and v are component rectangles. Zones xi, xii, xiii and x are spandrels that are the areal complements of the respective corner quadrants vii, viii, xi and vi.

r is the Corner Radius, the same for all four corners, d the "Calipered" Diagonal, and h the Zone i Diagonal.

The Castrum Area, A, is obviously the sum of the sub-rectangle and quadrant areas given by:-

$$A = [i] + [ii] + [iii] + [iv] + [v] + [vi] + [vii] + [viii] + [xi]$$

Equation 1

which can be re-stated as:-

$$\begin{aligned} A &= 2r(x-2r) + 2r(y-2r) + (x-2r)(y-2r) + \pi r^2 \\ &= xy - r^2(4-\pi) \end{aligned}$$

Equation 2

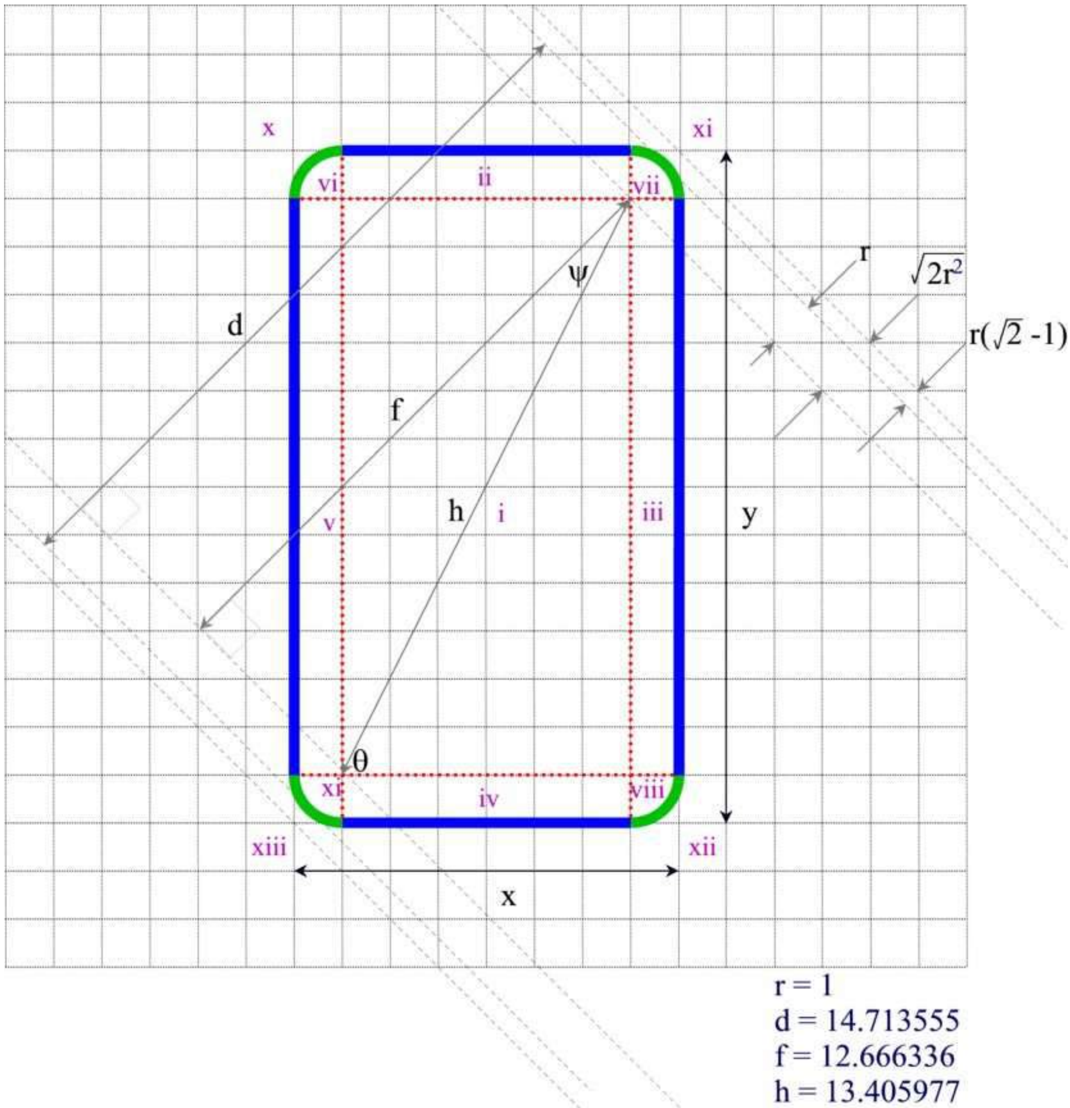

Figure One
The Lengths and Angles of
Castrum Mensuration

The Castrum Perimeter, P, is:-

$$P = 2[(x - 2r) + (y - 2r)] + 2\pi r$$
$$= 2[x + y + r(\pi - 4)]$$

Equation 3

and the Zone i Base Angle, θ, is given by:-

$$\theta = \tan^{-1}\left(\frac{y - 2ux}{x - 2ux}\right)$$

Equation 4

Inspection shows that the Zone i Calipered Base Angle, ψ, is given by:-

$$\psi = \theta - \frac{\pi}{4}$$

Equation 5

whilst the Zone i Diagonal is:-

$$h = \sqrt{(x - 2r)^2 + (y - 2r^2)}$$

Equation 6

At this juncture it is convenient to define the two auxiliaries. The Dimensionless Radius, u, is given by:-

$$u = \frac{r}{x}$$

Equation 7

or:-

$$r = ux$$

Equation 8

and Dimensionless Height (Aspect Ratio), v:-

$$v = \frac{y}{x}$$

Equation 9

or:-

$$y = vx$$

Equation 10

The use of the dimensionless auxiliaries u and v delays the divergence of The Ridders Method and facilitates the development of first-iterates as pseudoanalytic formulae.

Another virtue of u is that it confines the range of its possible radius-representative values to $0 \leq u \leq ½$. When u=0 the shape is a perfect rectangle and when u=0.5 the shape is a perfect circle. Intermediate values of u define castra.

The value of d and r is of course a function of Aspect Ratio also.

The use of u and v essentially frees our analysis of dependency upon scale or the units of measurement.

We may now proceed to develop convenient expressions of f and d in terms of x, u and v.

PART TWO
DEVELOPMENT

Permit that:-

$$k_1 = 1 + v^2$$

Equation 11

and:-

$$k_2 = v + 1$$

Equation 12

<u>The Development of Zone i Calipered Diagonal, f</u>

$$\begin{aligned} f &= h.\cos(\psi) \\ &= \sqrt{(x-2ux)^2 + (vx-2ux)^2}.\cos\left(\tan^{-1}\left(\frac{vx-2ux}{x-2ux}\right) - \frac{\pi}{4}\right) \\ &= x\sqrt{1+v^2+4u(2u-v-1)}.\cos\left(\tan^{-1}\left(\frac{vx-2ux}{x-2ux}\right) - \frac{\pi}{4}\right) \\ &= x\sqrt{1+v^2+4u(2u-v-1)}.\cos\left(\tan^{-1}\left(\frac{v-2u}{1-2u}\right) - \frac{\pi}{4}\right) \\ &= x\sqrt{k_1+4u(2u-k_2)}.\cos\left(\tan^{-1}\left(\frac{v-2u}{1-2u}\right) - \frac{\pi}{4}\right) \end{aligned}$$

Equation 13

<u>The Development of Measured Calipered Diagonal, d</u>

$$\begin{aligned} d &= 2ux + f \\ &= 2ux + x\sqrt{k_1+4u(2u-k_2)}.\cos\left(\tan^{-1}\left(\frac{v-2u}{1-2u}\right) - \frac{\pi}{4}\right) \\ &= x\left(2u + \sqrt{k_1+4u(2u-k_2)}.\cos\left(\tan^{-1}\left(\frac{v-2u}{1-2u}\right) - \frac{\pi}{4}\right)\right) \\ &= 2ux + x\sqrt{k_1+8u^2-4uk_2}.\sin\left(\tan^{-1}\left(\frac{2u-v}{2u-1}\right) - \frac{\pi}{4}\right) \end{aligned}$$

Equation 14

The square-rooted term of Equation Fourteen yields the form:-

$$d = 2ux + x\sqrt{8(u-R1)(u-R2)}.\sin\left(\tan^{-1}\left(\frac{2u-v}{2u-1}\right) - \frac{\pi}{4}\right)$$

Equation 15

where:

$$R1 = \frac{4(v+1) + \sqrt{16(v+1)^2 - 32(1+v^2)}}{16}$$

Equation 16

and:-

$$R2 = \frac{4(v+1) - \sqrt{16(v+1)^2 - 32(1+v^2)}}{16}$$

Equation 17

which may be expanded as:-

$$d = 2xu + x\sqrt{8\left(u - \frac{4k_2 + \sqrt{16k_2{}^2 - 32k_1}}{16}\right)\left(u - \frac{4k_2 - \sqrt{16k_2{}^2 - 32k_1}}{16}\right)}.\sin\left(\tan^{-1}\left(\frac{-v+2u}{-1+2u}\right) + \frac{\pi}{4}\right)$$

$$= 2xu + x.\left(\sqrt{8}\sqrt{\left(u - \frac{4k_2 + \sqrt{16k_2{}^2 - 32k_1}}{16}\right)\left(u - \frac{4k_2 - \sqrt{16k_2{}^2 - 32k_1}}{16}\right)}\right)\sin\left(\tan^{-1}\left(\frac{-v+2u}{-1+2u}\right) + \frac{\pi}{4}\right)$$

Equation 18

The simpler resumption of our development is, however:-

$$d = 2xu + x.\left(\sqrt{8}\sqrt{(u - R1)(u - R2)}\right)\sin\left(\tan^{-1}\left(\frac{-v+2u}{-1+2u}\right) + \frac{\pi}{4}\right)$$

$$= 2xu + x.\left(\sqrt{8}\sqrt{\left(u - \frac{4(v+1) + 4i(v-1)}{16}\right)\left(u - \frac{4(v+1) - 4i(v-1)}{16}\right)}\right)\sin\left(\tan^{-1}\left(\frac{-v+2u}{-1+2u}\right) + \frac{\pi}{4}\right)$$

$$= 2xu + x.\left(\sqrt{8}.\left(\frac{1}{4}\right)\sqrt{4u - v - 1 + i - iv}\sqrt{4u - v - 1 - i + iv}\right)\sin\left(\tan^{-1}\left(\frac{-v+2u}{-1+2u}\right) + \frac{\pi}{4}\right)$$

$$= 2xu + x.\left(\frac{\sqrt{2}}{2}\sqrt{4u - v - 1 + i - iv}\sqrt{4u - v - 1 - i + iv}\right)\sin\left(\tan^{-1}\left(\frac{-v+2u}{-1+2u}\right) + \frac{\pi}{4}\right)$$

$$= 2xu + x.\left(\frac{\sqrt{2}}{2}\sqrt{(4u - v - 1) + i(1-v)}\sqrt{(4u - v - 1) + i(v-1)}\right)\sin\left(\tan^{-1}\left(\frac{-v+2u}{-1+2u}\right) + \frac{\pi}{4}\right)$$

$$= 2xu + x.\left(\frac{\sqrt{2}}{2}\sqrt{(4u - v - 1)^2 + (1-v)^2}\right)\sin\left(\tan^{-1}\left(\frac{2u - v}{2u - 1}\right) + \frac{\pi}{4}\right)$$

$$= x\left(2u + \left(\frac{\sqrt{2}}{2}\sqrt{(4u - v - 1)^2 + (1-v)^2}\right)\sin\left(\tan^{-1}\left(\frac{2u - v}{2u - 1}\right) + \frac{\pi}{4}\right)\right)$$

Equation 19

<u>The First Numerical Solution Function, f(u)</u>

To facilitate the employment of programs of numerical differentiation, and in particular The Ridders Method, it is convenient now to define a Solution function f(u). Note that this relates geometrically to d rather than f.

As a preliminary we define the constant $k_{4,}$ the complement of Aspect Ratio, as:-

$$k_4 = 1 - v$$

Equation 20

By re-arrangement of the simplified equation for d, Equation Nineteen, we erect f(u) as:-

$$f(u) = \left\{ \left[2u + \frac{\sqrt{2}}{2} \sqrt{(4u - k_4)^2 + k_4^{\ 2}} \right] \sin\left(\tan^{-1}\left(\frac{2u - v}{2u - 1} \right) + \frac{\pi}{4} \right) \right\} - \frac{d}{x}$$

Equation 21

The value of f(u) is zero when, in the presence of a calipered diagonal defined by measurement, the value of u specifies the relevant r: I.e. when f(u) is a root of the equation for d.

Therefore, a sufficient solution of Equation Twenty-One leads directly to mensuration of the castrum.

PART THREE
THE USE OF CALIPERED DIAGONAL d
TO ESTABLISH FILLET RADIUS r
USING THE RIDDERS METHOD

The method due to CJF Ridders is a late twentieth-century scheme for the iterative numerical differentiation of equations and the determination of roots thereby. It is very rapidly convergent for certain kinds of mathematical structures.

The Ridders Method[1] needs to be presented with a continuous function, and three initial data that specify known point values.

The f(u) of Equation Twenty-One is algebraically-continuous, and in our context these starting values of u are convenient:-

$$u_1 = 0 \qquad \textbf{Equation 22a}$$
$$u_2 = 0.5 \qquad \textbf{Equation 22a}$$
$$u_3 = 0.25 \qquad \textbf{Equation 22a}$$

This is because zero and ½ are known extreme possibilities for u, whilst ¼ is their halfway position.

Effectively, our elaboration will always start at u=0.5 causing f(u) to diminish so that always $f(u_{n+1})<f(u_n)$.

Accordingly, the Ridders Criterion defined by:-

$$\operatorname{sgn}(xx, yy) = if\,((xx - yy) \geq 0{,}1{,}-1)$$

Condition 1

is always positive unity in our castrum mensuration situations.

The definition of The Ridders Method is:-

$$u_{n+3} = u_{n+2} + (u_{n+2} - u_n).\frac{(\operatorname{sgn}(f(u_n), f(u_{n+1}))).f(u_{n+2})}{\sqrt{f(u_{n+2})^2 - f(u_n).f(u_{n+1})}}$$

Equation 23

For which the first three iterates are clearly:-

$$u_4 = u_3 + (u_3 - u_1).\frac{(\operatorname{sgn}(f(u_1), f(u_2))).f(u_3)}{\sqrt{f(u_3)^2 - f(u_1).f(u_2)}} \qquad \textbf{Equation 24a}$$

$$u_5 = u_4 + (u_4 - u_2).\frac{(\operatorname{sgn}(f(u_2), f(u_3))).f(u_4)}{\sqrt{f(u_4)^2 - f(u_2).f(u_3)}} \qquad \textbf{Equation 24b}$$

$$u_6 = u_5 + (u_5 - u_3).\frac{(\operatorname{sgn}(f(u_3), f(u_4))).f(u_5)}{\sqrt{f(u_5)^2 - f(u_3).f(u_4)}} \qquad \textbf{Equation 24c}$$

These three points are potentially complex but usable values can be obtained as their complex moduli:-

$$|u_4| = \sqrt{\Re(u_4)^2 + \Im(u_4)^2}$$ **Equation 25a**

$$|u_5| = \sqrt{\Re(u_5)^2 + \Im(u_5)^2}$$ **Equation 25b**

$$|u_6| = \sqrt{\Re(u_6)^2 + \Im(u_6)^2}$$ **Equation 25c**

Now when the first iterate u_4 is computed it is evident from defined-radius experiments with a variety of u and v predicates that the said u_4 is accurate to at least twelve places of decimals and that no further iteration shall improve that.

Therefore, we may justifiably adopt Equation Twenty-Four-a as a pseudoanalytic formula for castrum radius in terms of aspect radius and calipered diagonal.

Since the sgn function is always +1 we may write:-

$$u_4 = \frac{1}{4} + \left(\frac{1}{4}\right) . \frac{f(u_3)}{\sqrt{f(u_3)^2 - f(u_1).f(u_2)}}$$

Equation 26

For notational convenience, the function $f_n \equiv f(u_n)$:-

$$u_4 = \frac{1}{4}\left[1 + \frac{f_3}{\sqrt{f_3^{\ 2} - f_1 f_2}}\right]$$

Equation 27

or:-

$$r = \frac{x}{4}\left[1 + \frac{f_3}{\sqrt{f_3^{\ 2} - f_1 f_2}}\right]$$

Equation 28

Algebraic Renditions for the Functions f_1, f_2 and f_3

To express f_1, f_2 and f_3 in computable forms we need to make relevant substitutions based upon the f(u) definitional Equation Twenty-One, as restated by Equation Twenty-Seven.

Simplification then gives convenient developments as:-

$$
\begin{aligned}
f_1 &= \frac{\sqrt{2}}{2}\sqrt{k_2{}^2 + k_4{}^2}.\sin\left(\tan^{-1}(v) + \frac{\pi}{4}\right) - \frac{d}{x} \\
&= \frac{\sqrt{2}}{2}\sqrt{(v+1)^2 + (1-v)^2}.\sin\left(\tan^{-1}(v) + \frac{\pi}{4}\right) - \frac{d}{x} \\
&= \frac{\sqrt{2}}{2}\sqrt{2\left(v^2+1\right)}.\sin\left(\tan^{-1}(v) + \frac{\pi}{4}\right) - \frac{d}{x} \\
&= \sqrt{v^2+1}.\sin\left(\tan^{-1}(v) + \frac{\pi}{4}\right) - \frac{d}{x}
\end{aligned}
$$

Equation 29

$$
\begin{aligned}
f_2 &= \left[1 + \left(\frac{\sqrt{2}}{2}\sqrt{(2-k_2)^2 + k_4{}^2}\right)\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right] - \frac{d}{x} \\
&= \left[1 + \left(\frac{\sqrt{2}}{2}\sqrt{(2-(v+1))^2 + (1-v)^2}\right)\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right] - \frac{d}{x} \\
&= \left[1 + \left(\frac{\sqrt{2}}{2}\sqrt{(1-v)^2 + (1-v)^2}\right)\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right] - \frac{d}{x} \\
&= \left[1 + \left(\frac{\sqrt{2}}{2}\sqrt{2(1-v)^2}\right)\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right] - \frac{d}{x} \\
&= \left[1 + \left(\frac{\sqrt{2}}{2}\sqrt{2}\sqrt{(1-v)^2}\right)\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right] - \frac{d}{x} \\
&= \left[1 + \sqrt{(1-v)^2}\,\sin\left(\frac{3}{4}\pi\right)\right] - \frac{d}{x} \\
&= \left[1 + (v-1)\sin\left(\frac{3}{4}\pi\right)\right] - \frac{d}{x}
\end{aligned}
$$

Equation 30

or:-

$$
f_2 = \frac{\sqrt{2}x(v-1) - 2(d-x)}{2x}
$$

Equation 31

$$
\begin{aligned}
f_3 &= \left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{(1-k_2)^2+k_4{}^2}\right\}.\sin\left\{\left(\tan^{-1}\left(\frac{0.5-v}{-0.5}\right)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x} \\
&= \left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{(1-(v+1))^2+(1-v)^2}\right\}.\sin\left\{\left(\tan^{-1}\left(\frac{0.5-v}{-0.5}\right)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x} \\
&= \left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{v^2+(1-v)^2}\right\}.\sin\left\{\left(\tan^{-1}\left(\frac{0.5-v}{-0.5}\right)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x} \\
&= \left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}\left(\frac{0.5-v}{-0.5}\right)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x} \\
&= \left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x}
\end{aligned}
$$

Equation 32

$$
\begin{aligned}
f_2{}^2 &= \left(\left[1+(v-1)\sin\left(\frac{3}{4}\pi\right)\right]-\frac{d}{x}\right).\left(\left[1+(v-1)\sin\left(\frac{3}{4}\pi\right)\right]-\frac{d}{x}\right) \\
&= \frac{1}{4}\frac{\left(-2x-\sqrt{2}xv+\sqrt{2}x+2d\right)^2}{x^2} \\
&= \frac{\left[2(d-x)-\sqrt{2}x(v-1)\right]^2}{4x^2}
\end{aligned}
$$

Equation 33

$$
f_1.f_2 = \left[\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right)-\frac{d}{x}\right].\left[\left\{1+(v-1).\sin\frac{3\pi}{4}\right\}-\frac{d}{x}\right]
$$

Equation 34

$$
f_3{}^2 = \left[\left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x}\right]^2
$$

Equation 35

The appropriate substitutions in Equation Twenty-Seven then give:-

$$
u_4 = \frac{1}{4}\left[1+\frac{\left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x}}{\sqrt{\left[\left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x}\right]^2-\left[\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right)-\frac{d}{x}\right].\left[\left\{1+(v-1).\sin\frac{3\pi}{4}\right\}-\frac{d}{x}\right]}}\right]
$$

Equation 36

$$r=\frac{x}{4}\left[1+\frac{\left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x}}{\sqrt{\left[\left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}\right]-\frac{d}{x}\right]^2-\left[\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right)-\frac{d}{x}\right].\frac{\sqrt{2}x(v-1)-2(d-x)}{2x}}}\right]$$

Equation 37

PART FOUR
FILLET RADIUS IN TERMS OF SHAPE CONSTANTS

For any particular castrum it is obvious that a unique fillet radius is the outcome of an interplay of constants. To assist computational simplifications and economies we can make the following equivalencies:-

$$c_1=\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}$$

Equation 38a

$$.c_2=\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right)$$

Equation 38b

$$c_3=1+(v-1)\sin\left(\frac{3}{4}\pi\right)$$

Equation 38c

$$c_4=\frac{d}{x}$$

Equation 38d

Notably, c_1, c_2 and c_3 are functions of the Aspect Ratio v only, whilst c_4 is a function of Width x and Diagonal d only. Relevant substitutions give:-

$$u_4=\frac{1}{4}\left[1+\frac{c_1-c_4}{\sqrt{(c_1-c_4)^2-(c_2-c_4)(c_3-c_4)}}\right]$$

Equation 39

which we may expand as:-

$$u_4=\frac{1}{4}\left[1+\frac{\sqrt{c_1{}^2-2c_1c_4-c_2c_3+c_2c_4+c_4c_3}+c_1-c_4}{\sqrt{c_1{}^2-2c_1c_4-c_2c_3+c_2c_4+c_4c_3}}\right]$$

Equation 40

Let us consolidate the square-rooted expression as:-

$$C = \sqrt{c_1^{\ 2} - 2c_1c_4 - c_2c_3 + c_2c_4 + c_4c_3}$$

Equation 41

Then $u_4 \approx r/x$ becomes:-

$$u_4 = \frac{1}{4}\left[\frac{C + c_1 - c_4}{C}\right]$$

Equation 42

or:-

$$u_4 = \frac{1}{4}\left[1 + \frac{c_1 - c_4}{C}\right]$$

Equation 43

Resolution of C

$$C = \sqrt{c_1^{\ 2} - 2c_1c_4 - c_2c_3 + c_2c_4 + c_4c_3}$$

Equation 41

or:-

$$C = \sqrt{c_1^{\ 2} - c_4\left(c_2 + c_3 - 2c_1\right) - c_2c_3}$$

Equation 44

Now:-

$$c_2 + c_3 - 2c_1 = 0$$

Equation 45

So:-

$$C = \sqrt{c_1^{\ 2} - c_2c_3}$$

Equation 46

The Components of C

$$c_1^{\ 2} = \left[\frac{1}{2} + \left\{\frac{\sqrt{2}}{2}\sqrt{1 - 2v(1 - v)}\right\}.\sin\left\{\left(\tan^{-1}(2v - 1) + \frac{\pi}{4}\right)\right\}\right]^2$$

Equation 47

which expands to:-

$$c_1^{\,2}=\left[\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}\right]^2$$
$$=\frac{3}{4}+\frac{1}{2}\sqrt{2}\sqrt{1-2v+2v^2}.\sin\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)-\cos\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)^2.\left(\frac{1}{2}-v+v^2\right)+v^2-v$$
$$=\frac{3}{4}+\frac{1}{2}\sqrt{2}\sqrt{1-2v+2v^2}.\sin\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)-\cos\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)^2.\left[\left(v-\frac{1}{2}\right)^2+\frac{1}{4}\right]+v^2-v$$

Equation 48

Furthermore:-

$$c_2c_3=\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right).\left[1+(v-1)\sin\left(\frac{3}{4}\pi\right)\right]$$
$$=\sqrt{v^2-1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right).\left[1+\frac{\sqrt{2}}{2}v-\frac{\sqrt{2}}{2}\right]$$

Equation 49

Therefore, deleting the null term in c_4:-

$$C=\sqrt{c_1{}^2-c_2c_3}$$
$$=\sqrt{\frac{3}{4}+\frac{\sqrt{2}}{2}\sqrt{1-2v+2v^2}.\sin\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)-\cos\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)^2.\left[\left(v-\frac{1}{2}\right)^2+\frac{1}{4}\right]+v^2-v-\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right).\left[1+\frac{\sqrt{2}}{2}v-\frac{\sqrt{2}}{2}\right]}$$

Equation 50

From which appropriate substitutions give:-

$$r=\frac{x}{4}\left[1+\frac{\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}-\frac{d}{x}}{\sqrt{\frac{3}{4}+\frac{\sqrt{2}}{2}\sqrt{1-2v+2v^2}.\sin\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)-\cos\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)^2.\left[\left(v-\frac{1}{2}\right)^2+\frac{1}{4}\right]+v^2-v-\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right).\left[1+\frac{\sqrt{2}}{2}v-\frac{\sqrt{2}}{2}\right]}}\right]$$

Equation 51

$$r=\frac{x}{4}\left[1+\frac{\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}-\frac{d}{x}}{\sqrt{\frac{3}{4}+\frac{\sqrt{2}}{2}\sqrt{1-2v+2v^2}.\sin\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)-\cos\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)^2.\left[v^2-v+\frac{1}{2}\right]+v^2-v-\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right).\left[1+\frac{\sqrt{2}}{2}v-\frac{\sqrt{2}}{2}\right]}}\right]$$

Equation 52

PART FIVE
APPROACHES FROM
THE FIGURE DIAGONAL LAMBDA
USING MINOR METHODS

The Preparation of Analytical Solution Functions for Numerical Use

The Castrum Lineal Diagonal, λ, is a straight-line diagonal across the whole castrum and intersecting opposite Zone i corners. It is thus defined by:-

$$\begin{aligned}\lambda &= h - 2r \\ &= \sqrt{(x-2r)^2 + (y-2r)^2} + 2r \\ &= \sqrt{x^2 + y^2 - 4r(x+y) + 8r^2} + 2r\end{aligned}$$

Equation 53

In terms of shape dimensions solution functions g(r) and g(u) can be defined as below:-

$$g(r) = \sqrt{(x-2r)^2 + (y-2r)^2} + 2r - \lambda$$

Equation 54

$$g(u) = \sqrt{(x-2ux)^2 + (y-2ux)^2} + 2ux - \lambda$$

Equation 55

An interesting first differential of g(u) with respect to u is:-

$$\begin{aligned}\frac{dg}{du}(u) &= \frac{-4x^2 + 16ux^2 - 4yx}{2\sqrt{x^2 - 4ux^2 + 8u^2x^2 + y^2 - 4yux}} + 2x \\ &= 2\left[x - \frac{x^2(1-4u) + yx}{\sqrt{x^2 + y^2 - 4ux\{x(1-2u) + y\}}}\right]\end{aligned}$$

Equation 56

Trials have demonstrated that Inverse Quadratic Interpolation cannot yield an iterate definitive of radius.

Both Muller's Method and Newton's Method do however.

Muller's Method

In this application we may define Muller's Method in the following way:-

$$w_1 = 0$$ **Equation 57a**

$$w_2 = 0.25$$ **Equation 57b**

$$w_3 = 0.5$$ **Equation 57c**

$$w_4 = w_1 \frac{w_2 w_3}{[g(w_1) - g(w_2)][g(w_1) - g(w_3)]} + w_2 \frac{w_1 w_3}{[g(w_2) - g(w_1)][g(w_2) - g(w_3)]} + w_3 \frac{w_1 w_2}{[g(w_3) - g(w_1)][g(w_3) - g(w_2)]}$$

Equation 58

$$\Xi 2(x, y) = \frac{g(x)}{x - y} + \frac{g(y)}{y - x}$$

Equation 59

$$\Xi 3(x, y, z) = \frac{g(x)}{(x - y)(x - z)} + \frac{g(y)}{(y - x)(y - z)} + \frac{g(z)}{(z - x)(z - y)}$$

Equation 60

$$W_n = \Xi 2(w_n, w_{n-1}) + \Xi 2(w_n, w_{n-2}) - \Xi 2(w_{n-1}, w_{n-2})$$

Equation 61

$$w_n = w_{n-1} - \frac{2g(w_{n-1})}{W_{n-1} - \sqrt{W_{n-1}{}^2 - 4g(w_{n-1}).\Xi 3(w_{n-1}, w_{n-2}, w_{n-3})}}$$

Equation 62

The points w are of course directly analogous to u but in contrast to The Ridders Method we have to wait until w_7 to obtain eight-figure accuracy in the radius estimator w.

Muller's Method is therefore unlikely to be computationally profitable.

Newton's Method

Newton's Method is defined by:-

$$u_{n+1} = u_n - \frac{g(u_n)}{\frac{dg}{du}(u_n)}$$

Equation 63

and achieves better than twelve-figure accuracy by the sixth iterate u_6. Furthermore, it only requires one starter point defined as $u_1 = 0.25$.

As an iterative program it may be economically competitive with The Ridders Method in this application.

Reference

1 "A New Algorithm for Computing a Single Root of a Real Continuous Function"
CJF Ridders
IEEE Transactions on Circuits and Systems
Vol. CAS-26. No. 11 November 1979
pp 979-980

APPENDIX A
CASTRUM DIAGAMS
FOR TRIAL COMPUTATIONS

Figure Number	Grid Dimensions in Unit Squares		Grid Dimensions in Millimeters		Mean Square Side Length in Millimeters	Figure Diagonals in Millimeters			Prior Metrics			Figure Diagonals in Centimeters		
	H	**V**	**Width**	**Height**		**d**	**f**	**h**	**r**	**x**	**y**	**d**	**f**	**h**
One	20	20	151.2	151.8	7.575	111.4	95.9	101.5	1	8	14	14.713555	12.666336	13.405977
Two	20	20	151.3	151.8	7.5775	107	80.1	86.3	1.773	8	14	14.132409	10.579495	11.398382
Three	20	20	151	151.6	7.565	99.1	53.9	62.4	3	8	14	13.088988	7.119036	8.241704
Four	20	20	151.1	151.6	7.5675	77.9	32.3	34.1	3	8	10	10.288922	4.266138	4.503880
μ					7.57125									
σ					0.00515									
CV%					0.06807									

Table A1
Source Data for Castrum Geometry

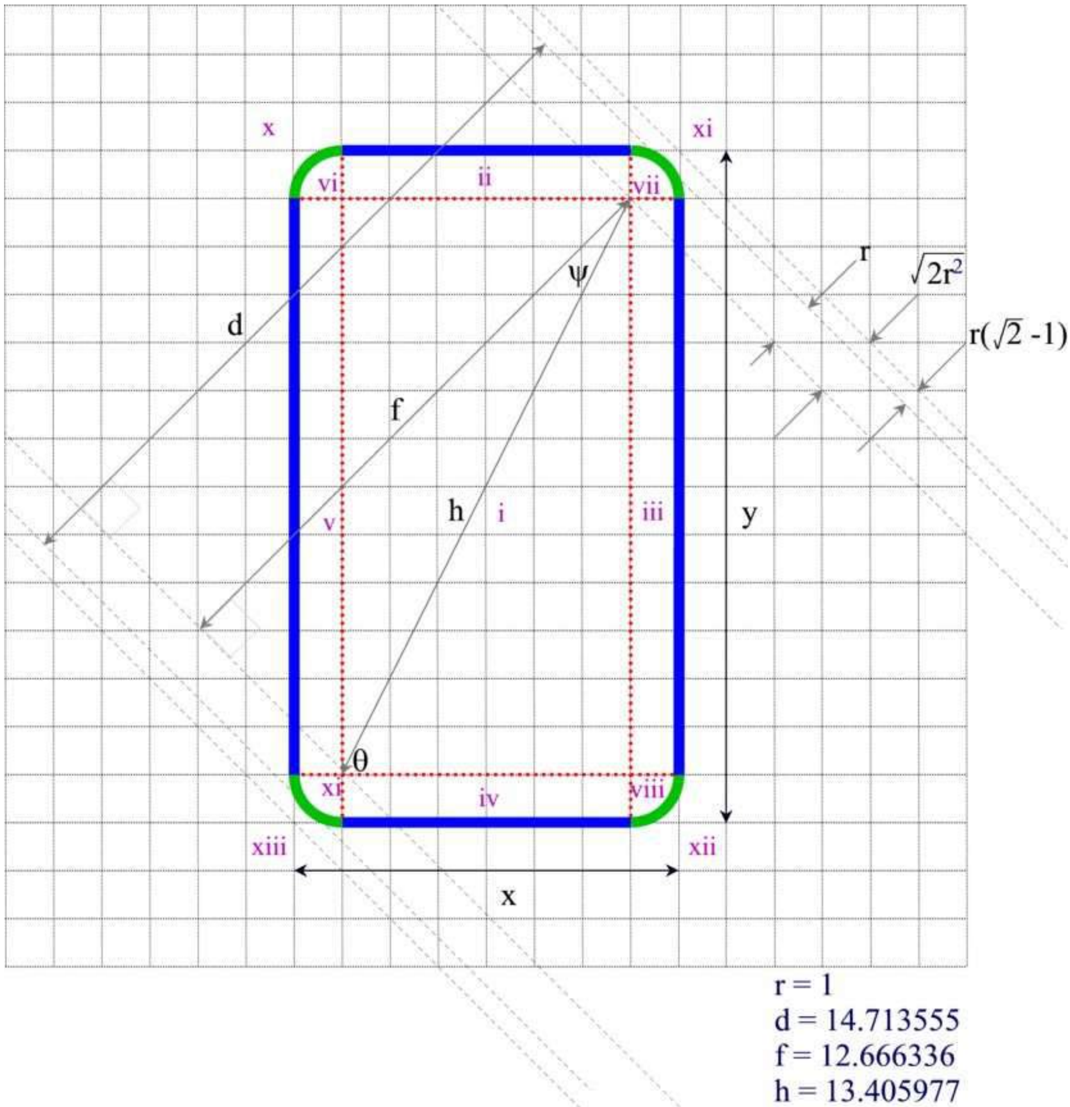

Figure A1
Castrum Geometry for
r = 1; x = 8; y = 14
d, f and h are as measured from paper

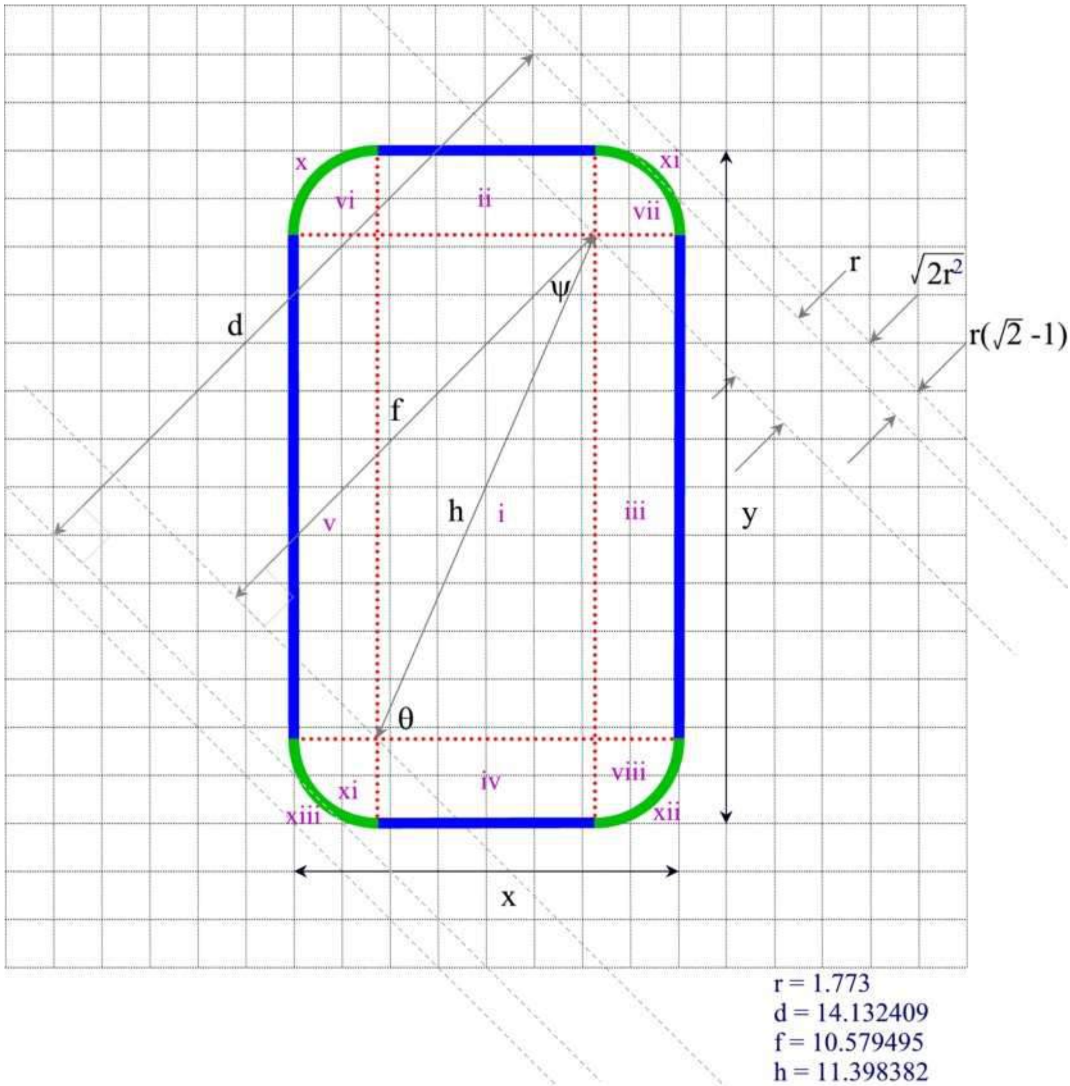

Figure A2
Castrum Geometry for
r = 1.773; x = 8; y = 14
d, f and h are as measured from paper

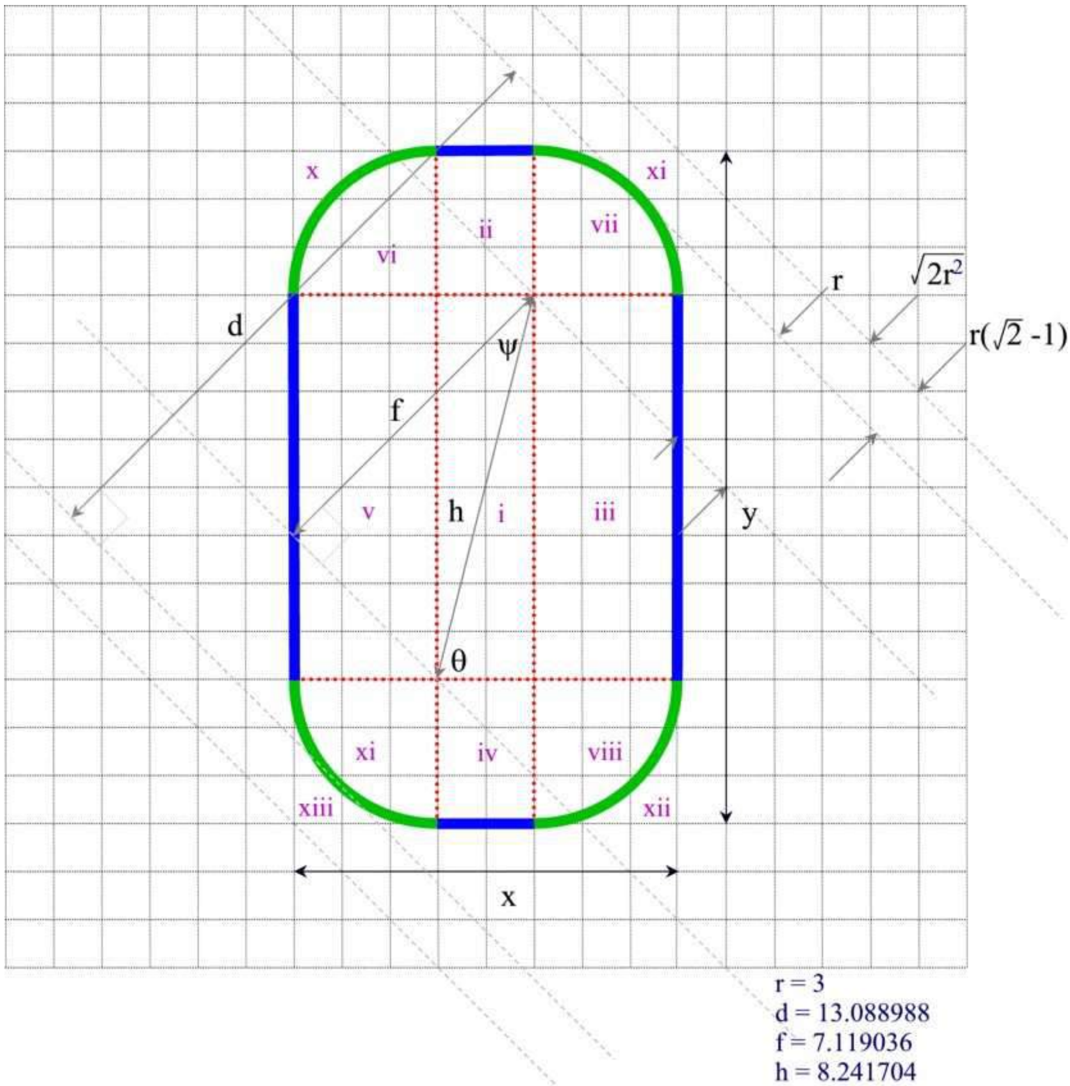

Figure A3
Castrum Geometry for
r = 3; x = 8; y = 14
d, f and h are as measured from paper

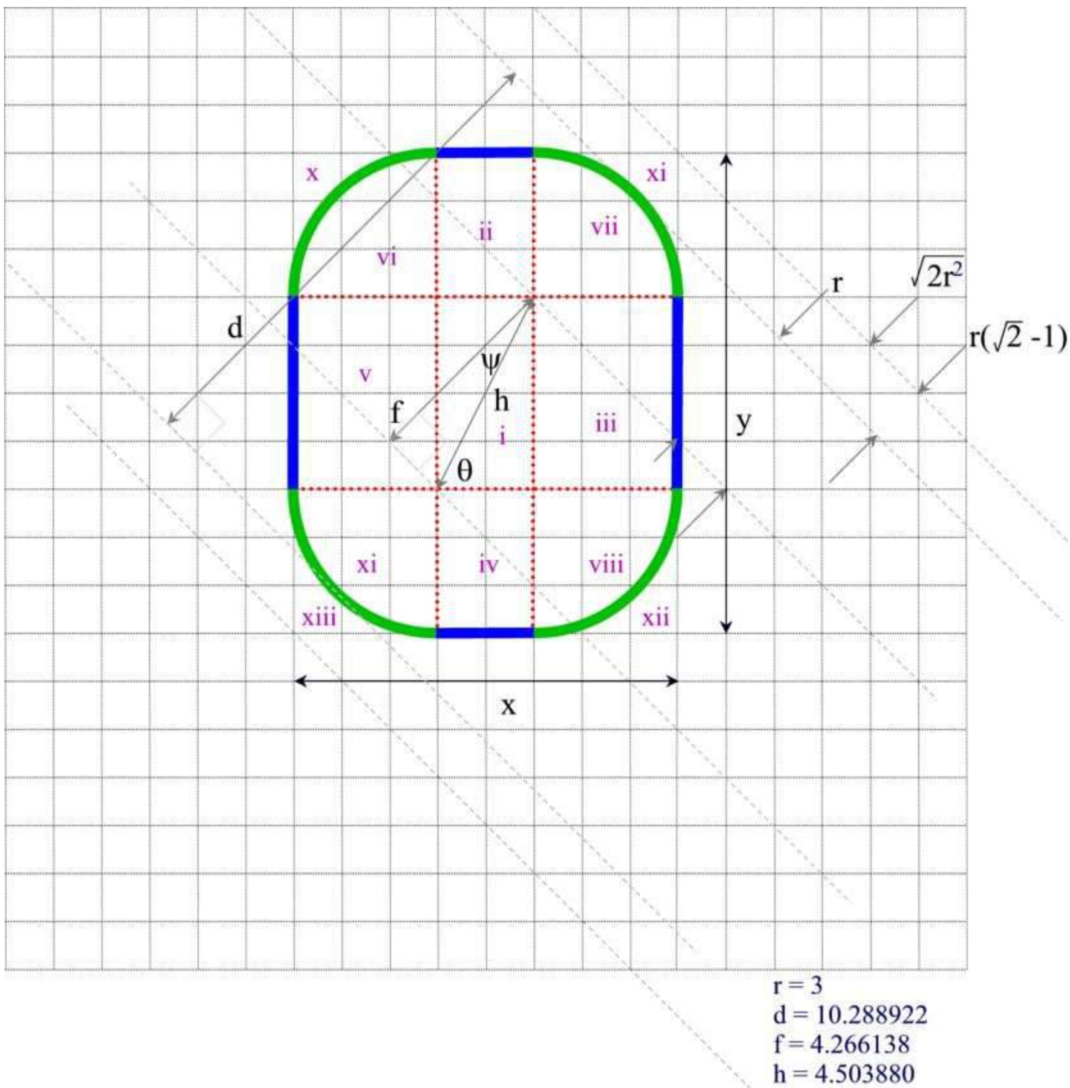

Figure A4
Castrum Geometry for
r = 3; x = 8; y = 10
d, f and h are as measured from paper

APPENDIX B
CASTRUM EXCEL® WORKSHEETS
FOR TRIAL COMPUTATIONS

NAME:	CASTRUM Figure One	
Datum	**Symbol**	**Value**
Width	x	8
Height	y	14
Measured Corner Radius	r_m	1
Measured Calipered Diagonal	d_m	14.71355456
Zone i Diagonal	h	13.41640786
Spandrel Radius Compliment	w	0.414213562
Approximate Castrum Lineal Diagonal	λ_{approx}	15.41640786
Castrum Area Constant	k_A	0.858407346
Prior Castrum Area	A_{pr}	111.1415927
Prior Castrum Perimeter	P_{pr}	42.28318531
Dimensionless Radius	u	0.125
Dimensionless Height	v	1.75
Trailing v Constant	k_1	4.0625
Leading v Constant	k_2	2.75
v Constant	k_3	1.75
Unity minus v	k_4	-0.75
Zone i Base Angle	θ	1.107148718
Zone i Calipered Height Angle	ψ	0.321750554
Zone i Calipered Diagonal	f	12.72792206
Castrum Calipered Diagonal	d	14.72792206
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Castrum Lambda Root Function	$g_\lambda(u_1)$	0.708107632
Second Castrum Lambda Root Function	$g_\lambda(u_2)$	-1.416407865
Third Castrum Lambda Root Function	$g_\lambda(u_3)$	-0.646078251
Pseudoanalytic Component Term One	c_1	1.737436867
Pseudoanalytic Component Term Two	c_2	1.944543648
Pseudoanalytic Component Term Three	c_3	1.530330086
Pseudoanalytic Component Term Four	c_4	1.839194321
Consolidated Component Term	C	0.207106781
First Ridder Root (Actual Dimensionless Radius)	u_4	0.127167888
Actual Corner Radius according to Pseudoanalytic Formula	r_r	1.017343102
Prior Castrum Area	A_{pr}	111.1415927
Posterior Castrum Area	A_{po}	111.1115596
Prior Castrum Perimeter	P_{pr}	42.28318531
Posterior Castrum Perimeter	P_{po}	42.25341041
Area Specific Defect (Percentage)	SD_A	-0.027022366
Perimeter Specific Defect (Percentage)	SD_P	-0.07041781

Table B1
Castrum Trial Data and Results for
Figure One

NAME:	CASTRUM Figure Two	
Datum	**Symbol**	**Value**
Width	x	8
Height	y	14
Measured Corner Radius	r_m	1.773
Measured Calipered Diagonal	d_m	14.13240878
Zone i Diagonal	h	11.36328438
Spandrel Radius Compliment	w	0.734400646
Approximate Castrum Lineal Diagonal	λ_{approx}	14.90928438
Castrum Area Constant	k_A	0.858407346
Prior Castrum Area	A_{pr}	109.3015716
Prior Castrum Perimeter	P_{pr}	40.95608755
Dimensionless Radius	u	0.221625
Dimensionless Height	v	1.75
Trailing v Constant	k_1	4.0625
Leading v Constant	k_2	2.75
v Constant	k_3	1.75
Unity minus v	k_4	-0.75
Zone i Base Angle	θ	1.168030722
Zone i Calipered Height Angle	ψ	0.382632558
Zone i Calipered Diagonal	f	10.54154789
Castrum Calipered Diagonal	d	14.08754789
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Castrum Lambda Root Function	$g_\lambda(u_1)$	1.215231112
Second Castrum Lambda Root Function	$g_\lambda(u_2)$	-0.909284384
Third Castrum Lambda Root Function	$g_\lambda(u_3)$	-0.13895477
Pseudoanalytic Component Term One	c_1	1.737436867
Pseudoanalytic Component Term Two	c_2	1.944543648
Pseudoanalytic Component Term Three	c_3	1.530330086
Pseudoanalytic Component Term Four	c_4	1.766551098
Consolidated Component Term	C	0.207106781
First Ridder Root (Actual Dimensionless Radius)	u_4	0.214856015
Actual Corner Radius according to Pseudoanalytic Formula	r_r	1.718848116
Prior Castrum Area	A_{pr}	109.3015716
Posterior Castrum Area	A_{po}	109.463888
Prior Castrum Perimeter	P_{pr}	40.95608755
Posterior Castrum Perimeter	P_{po}	41.0490563
Area Specific Defect (Percentage)	SD_A	0.148503241
Perimeter Specific Defect (Percentage)	SD_P	0.226996169

Table B2
Castrum Trial Data and Results for
Figure Two

NAME:	CASTRUM Figure Three	
Datum	**Symbol**	**Value**
Width	x	8
Height	y	14
Measured Corner Radius	r_m	3
Measured Calipered Diagonal	d_m	13.088988
Zone i Diagonal	h	8.246211251
Spandrel Radius Compliment	w	1.242640687
Approximate Castrum Lineal Diagonal	λ_{approx}	14.24621125
Castrum Area Constant	k_A	0.858407346
Prior Castrum Area	A_{pr}	104.2743339
Prior Castrum Perimeter	P_{pr}	38.84955592
Dimensionless Radius	u	0.375
Dimensionless Height	v	1.75
Trailing v Constant	k_1	4.0625
Leading v Constant	k_2	2.75
v Constant	k_3	1.75
Unity minus v	k_4	-0.75
Zone i Base Angle	θ	1.325817664
Zone i Calipered Height Angle	ψ	0.5404195
Zone i Calipered Diagonal	f	7.071067812
Castrum Calipered Diagonal	d	13.07106781
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Castrum Lambda Root Function	$g_\lambda(u_1)$	1.878304245
Second Castrum Lambda Root Function	$g_\lambda(u_2)$	-0.246211251
Third Castrum Lambda Root Function	$g_\lambda(u_3)$	0.524118363
Pseudoanalytic Component Term One	c_1	1.737436867
Pseudoanalytic Component Term Two	c_2	1.944543648
Pseudoanalytic Component Term Three	c_3	1.530330086
Pseudoanalytic Component Term Four	c_4	1.636123493
Consolidated Component Term	C	0.207106781
First Ridder Root (Actual Dimensionless Radius)	u_4	0.37229606
Actual Corner Radius according to Pseudoanalytic Formula	r_r	2.978368482
Prior Castrum Area	A_{pr}	104.2743339
Posterior Castrum Area	A_{po}	104.3853441
Prior Castrum Perimeter	P_{pr}	38.84955592
Posterior Castrum Perimeter	P_{po}	38.88669323
Area Specific Defect (Percentage)	SD_A	0.106459806
Perimeter Specific Defect (Percentage)	SD_P	0.095592617

Table B3
Castrum Trial Data and Results for
Figure Three

NAME:	CASTRUM Figure Four	
Datum	**Symbol**	**Value**
Width	x	8
Height	y	10
Measured Corner Radius	r_m	3
Measured Calipered Diagonal	d_m	10.28892191
Zone i Diagonal	h	4.472135955
Spandrel Radius Compliment	w	1.242640687
Approximate Castrum Lineal Diagonal	λ_{approx}	10.47213595
Castrum Area Constant	k_A	0.858407346
Prior Castrum Area	A_{pr}	72.27433388
Prior Castrum Perimeter	P_{pr}	30.84955592
Dimensionless Radius	u	0.375
Dimensionless Height	v	1.25
Trailing v Constant	k_1	2.5625
Leading v Constant	k_2	2.25
v Constant	k_3	1.25
Unity minus v	k_4	-0.25
Zone i Base Angle	θ	1.107148718
Zone i Calipered Height Angle	ψ	0.321750554
Zone i Calipered Diagonal	f	4.242640687
Castrum Calipered Diagonal	d	10.24264069
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Castrum Lambda Root Function	$g_\lambda(u_1)$	2.33411252
Second Castrum Lambda Root Function	$g_\lambda(u_2)$	-0.472135955
Third Castrum Lambda Root Function	$g_\lambda(u_3)$	0.738966596
Pseudoanalytic Component Term One	c_1	1.383883476
Pseudoanalytic Component Term Two	c_2	1.590990258
Pseudoanalytic Component Term Three	c_3	1.176776695
Pseudoanalytic Component Term Four	c_4	1.286115239
Consolidated Component Term	C	0.207106781
First Ridder Root (Actual Dimensionless Radius)	u_4	0.368016703
Actual Corner Radius according to Pseudoanalytic Formula	r_r	2.944133624
Prior Castrum Area	A_{pr}	72.27433388
Posterior Castrum Area	A_{po}	72.55939139
Prior Castrum Perimeter	P_{pr}	30.84955592
Posterior Castrum Perimeter	P_{po}	30.94546814
Area Specific Defect (Percentage)	SD_A	0.394410432
Perimeter Specific Defect (Percentage)	SD_P	0.310903067

Table B4
Castrum Trial Data and Results for
Figure Four

NAME:	CASTRUM Unit Circle	
Datum	**Symbol**	**Value**
Width	x	1
Height	y	1
Measured Corner Radius	r_m	0.5
Measured Calipered Diagonal	d_m	1
Zone i Diagonal	h	0
Spandrel Radius Compliment	w	0.207106781
Approximate Castrum Lineal Diagonal	λ_{approx}	1
Castrum Area Constant	k_A	0.858407346
Prior Castrum Area	A_{pr}	0.785398163
Prior Castrum Perimeter	P_{pr}	3.141592654
Dimensionless Radius	u	0.5
Dimensionless Height	v	1
Trailing v Constant	k_1	2
Leading v Constant	k_2	2
v Constant	k_3	1
Unity minus v	k_4	0
Zone i Base Angle	θ	#DIV/0!
Zone i Calipered Height Angle	ψ	#DIV/0!
Zone i Calipered Diagonal	f	#DIV/0!
Castrum Calipered Diagonal	d	#DIV/0!
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Castrum Lambda Root Function	$g_\lambda(u_1)$	0.414213562
Second Castrum Lambda Root Function	$g_\lambda(u_2)$	8.28426E-11
Third Castrum Lambda Root Function	$g_\lambda(u_3)$	0.207106781
Pseudoanalytic Component Term One	c_1	1.207106781
Pseudoanalytic Component Term Two	c_2	1.414213562
Pseudoanalytic Component Term Three	c_3	1
Pseudoanalytic Component Term Four	c_4	1
Consolidated Component Term	C	0.207106781
First Ridder Root (Actual Dimensionless Radius)	u_4	0.5
Actual Corner Radius according to Pseudoanalytic Formula	r_r	0.5
Prior Castrum Area	A_{pr}	0.785398163
Posterior Castrum Area	A_{po}	0.785398163
Prior Castrum Perimeter	P_{pr}	3.141592654
Posterior Castrum Perimeter	P_{po}	3.141592654
Area Specific Defect (Percentage)	SD_A	0
Perimeter Specific Defect (Percentage)	SD_P	0

Table B5
Castrum Trial Data and Results for
The Unit Circle

NAME:	CASTRUM Unit Square	
Datum	**Symbol**	**Value**
Width	x	1
Height	y	1
Measured Corner Radius	r_m	0
Measured Calipered Diagonal	d_m	1.414213562
Zone i Diagonal	h	1.414213562
Spandrel Radius Compliment	w	0
Approximate Castrum Lineal Diagonal	λ_{approx}	1.414213562
Castrum Area Constant	k_A	0.858407346
Prior Castrum Area	A_{pr}	1
Prior Castrum Perimeter	P_{pr}	4
Dimensionless Radius	u	0
Dimensionless Height	v	1
Trailing v Constant	k_1	2
Leading v Constant	k_2	2
v Constant	k_3	1
Unity minus v	k_4	0
Zone i Base Angle	θ	0.785398163
Zone i Calipered Height Angle	ψ	0
Zone i Calipered Diagonal	f	1.414213562
Castrum Calipered Diagonal	d	1.414213562
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Castrum Lambda Root Function	$g_\lambda(u_1)$	0
Second Castrum Lambda Root Function	$g_\lambda(u_2)$	-0.414213562
Third Castrum Lambda Root Function	$g_\lambda(u_3)$	-0.207106781
Pseudoanalytic Component Term One	c_1	1.207106781
Pseudoanalytic Component Term Two	c_2	1.414213562
Pseudoanalytic Component Term Three	c_3	1
Pseudoanalytic Component Term Four	c_4	1.414213562
Consolidated Component Term	C	0.207106781
First Ridder Root (Actual Dimensionless Radius)	u_4	-6.66134E-16
Actual Corner Radius according to Pseudoanalytic Formula	r_r	0
Prior Castrum Area	A_{pr}	1
Posterior Castrum Area	A_{po}	1
Prior Castrum Perimeter	P_{pr}	4
Posterior Castrum Perimeter	P_{po}	4
Area Specific Defect (Percentage)	SD_A	0
Perimeter Specific Defect (Percentage)	SD_P	0

Table B6
Castrum Trial Data and Results for
The Unit Square