A Simple and Non-Trigonometric Development of Castrum Mensuration

by

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Those who studied my 3 November 2012 submission entitled "Pseudoanalytic Mensuration of the Castrum by Means of The Ridders Method" may recall a large and complicated formula that I proposed to compute the Corner Radius (Fillet Radius) of a castrum.

For our convenience I reproduce it here:-

$$r=\frac{x}{4}\left[1+\frac{\frac{1}{2}+\left\{\frac{\sqrt{2}}{2}\sqrt{1-2v(1-v)}\right\}.\sin\left\{\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)\right\}-\frac{d}{x}}{\sqrt{\frac{3}{4}+\frac{\sqrt{2}}{2}\sqrt{1-2v+2v^2}.\sin\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)-\cos\left(\tan^{-1}(2v-1)+\frac{\pi}{4}\right)^2.\left[v^2-v+\frac{1}{2}\right]+v^2-v-\sqrt{v^2+1}.\sin\left(\tan^{-1}(v)+\frac{\pi}{4}\right).\left[1+\frac{\sqrt{2}}{2}v-\frac{\sqrt{2}}{2}\right]}}\right]$$

Equation 52
Ex CASTRUM.DOC

In these days of microcomputing we have become prodigal of power, but we need to remember that every operation has penalties in time, capacity and cost.

Including as they do nests of cardinal operations functional solutions are especially extravagant. Equation Fifty-Two from the prior paper includes four Arctangents; three Sines; one Cosine; and no less than eight Square Roots of which four are merely $2^{½}$. I have not bothered to count the cardinal operations, which are clearly many.

We desire to derive an equivalent formula for Corner Radius that minimises the count of operations, especially expensive ones, without compromising accuracy.

Thus we may economise great systems by maximising the alacrity of their minimum parts.

PART ONE
DEFINITION

There is a shape on the Euclidean plane that is like a rectangle with quadrant corners. I do not know the conventional name of this shape. Some people call it a playing-card shape, and some a filleted rectangle. I shall call it a castrum, the Roman word for an encampment, because the figure suggests the usual plan of such a structure.

Accurate mensuration of such a shape is problematical, because whilst theoretically fixed by knowledge of three points, quadrant radius is difficult to measure.

Therefore, an indirect but accurate method for the determination of fillet radius is desirable.

The method I present depends upon three linear data only: x, the Width of the castrum measured normal to its vertical straight sides; y, the Height of the castrum measured normal to its horizontal straight sides; and d, the “Calipered” Diagonal, a cross lineament parallel to the bisectors of opposite quadrants, between the extreme tangents of those quadrants.

Figure One, in which the blue and green circuit delineates the castrum, illustrates definitional lines and angles. In the context of this diagram, and the other figure diagrams of Appendix A it should be noted that whilst the radius value is definitional, the d, f and h values noted in blue are computed from measured paper and are of only three-figure significance.

The areal zones i, ii, iii, iv and v are component rectangles. Zones xi, xii, xiii and x are spandrels that are the areal complements of the respective corner quadrants vii, viii, xi and vi.

r is the Corner Radius, the same for all four corners, d the “Calipered” Diagonal, and h the Zone i Diagonal.

The Castrum Area, A, is obviously the sum of the sub-rectangle and quadrant areas given by:-

$$A = [i] + [ii] + [iii] + [iv] + [v] + [vi] + [vii] + [viii] + [xi]$$

Equation 1

which can be re-stated as:-

$$\begin{aligned} A &= 2r(x-2r) + 2r(y-2r) + (x-2r)(y-2r) + \pi r^2 \\ &= xy - r^2(4-\pi) \end{aligned}$$

Equation 2

As in the previous study it is convenient to define two auxiliaries. The Dimensionless Radius, u, is given by:-

$$u = \frac{r}{x}$$

Equation 3

or:-

$$r = ux$$

Equation 4

and Dimensionless Height (Aspect Ratio), v:-

$$v = \frac{y}{x}$$

Equation 5

or:-

$$y = vx$$
Equation 6

The use of the dimensionless auxiliaries u and v delays the divergence of The Ridders Method and facilitates the development of first-iterates as pseudoanalytic formulae.

Another virtue of u is that it confines the range of its possible radius-representative values to $0 \le u \le ½$. When u=0 the shape is a perfect rectangle and when u=0.5 the shape is a perfect circle. Intermediate values of u define castra.

The value of d and r is of course a function of Aspect Ratio also.

The use of u and v essentially frees our analysis of dependency upon scale or the units of measurement, though in this new simplified treatment u and v (especially v) shall play secondary roles..

In this treatment we also need to consider the Internal (i.e. Zone i) Castrum Width, a, defined as:-

$$a = x - 2ux$$
Equation 7

and Internal Castrum Height, b, given by:-

$$b = y - 2ux$$
Equation 8

Furthermore, the Internal Aspect Ratio, ρ, is specified by:-

$$\rho = \frac{b}{a}$$
Equation 9

These new lengths, a and b, have been added in orange to the definitional diagram Figure One.

The Castrum Perimeter, P, is:-

$$\begin{aligned} P &= 2\left[(x-2r)+(y-2r)\right]+2\pi r \\ &= 2\left[x+y+r(\pi-4)\right] \end{aligned}$$
Equation 10

and the Zone i Base Angle, θ, is given by:-

$$\theta = \tan^{-1}\left(\frac{y-2ux}{x-2ux}\right) = \tan^{-1}\left(\frac{vx-2ux}{x-2ux}\right) = \tan^{-1}\left(\frac{x(v-2u)}{x(1-2u)}\right) = \tan^{-1}\left(\frac{v-2u}{1-2u}\right)$$
Equation 11

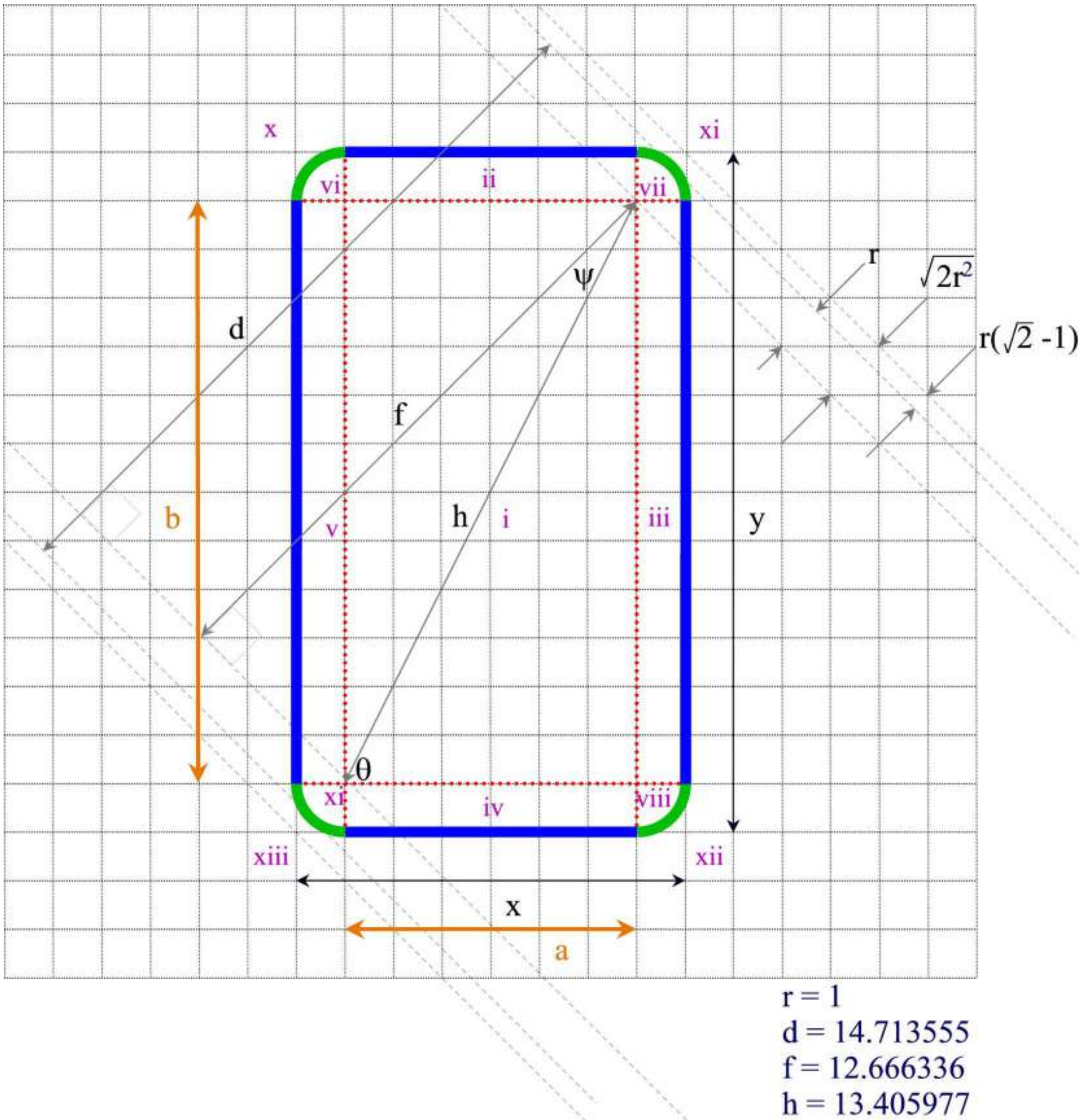

Figure One
The Lengths and Angles of
Castrum Mensuration

Inspection shows that the Zone i Calipered Base Angle, ψ, is given by:-

$$\psi = \theta - \frac{\pi}{4} = \theta - \phi$$

Equation 12

where ϕ is of course π/4 radians.
The Zone i Diagonal is:-

$$h = \sqrt{(x-2r)^2 + \left(y-2r^2\right)} = \sqrt{a^2+b^2}$$

Equation 13

PART TWO
ANALYTIC GEOMETRY

We may now proceed to develop convenient expressions of f and d in terms of a, b and ρ.

The Development of Zone i Calipered Diagonal, f

As in my previous study "Pseudoanalytic Mensuration of the Castrum by Means of The Ridders Method" f, the Zone i Calipered Diagonal, is given by:-

$$f = h.\cos(\psi)$$

Equation 14

Key to our simplified treatment are these identities, where α and β are arbitrary angles[1]:-

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Equation 15

and[2]:-

$$\sin\left(\tan^{-1}\alpha\right) = \frac{\alpha}{\sqrt{1+\alpha^2}}$$ **Equation 16a**

$$\cos\left(\tan^{-1}\alpha\right) = \frac{1}{\sqrt{1+\alpha^2}}$$ **Equation 16b**

Equations Sixteen a and Sixteen b are critical to the elimination of trigonometric functions from the developments.

The employment of these identities allows us to develop f as:-

$$\begin{aligned} f &= \sqrt{a^2+b^2}.\cos(\theta-\phi) \\ &= \sqrt{a^2+b^2}.(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ &= \sqrt{a^2+b^2}.\left(\cos\theta\frac{1}{\sqrt{2}} + \sin\theta\frac{1}{\sqrt{2}}\right) \\ &= \frac{\sqrt{a^2+b^2}}{\sqrt{2}}.(\cos\theta + \sin\theta) \\ &= \frac{\sqrt{a^2+b^2}}{\sqrt{2}}.\left(\frac{1}{\sqrt{1+\rho^2}} + \frac{\rho}{\sqrt{1+\rho^2}}\right) \\ &= \frac{\sqrt{a^2+b^2}}{\sqrt{2}\sqrt{1+\rho^2}}.(1+\rho) \end{aligned}$$

Equation 17

We may note in passing that the square of f is given by:-

$$f^2 = \sqrt{\frac{a^2 + b^2}{2\left(1+\rho^2\right)}\left(1+\rho\right)^2}$$

Equation 18

In f, substitution for a, b and ρ gives:-

$$f = \sqrt{\frac{(x-2ux)^2 + (y-2ux)^2}{2\left[1+\left(\frac{y-2ux}{x-2ux}\right)^2\right]}.\left[1+\left(\frac{y-2ux}{x-2ux}\right)\right]^2}$$

Equation 19

The Development of Measured Calipered Diagonal, d

The Measured Calipered Diagonal, d, is delineated parallel to the bisectors of opposite fillet quadrants and accordingly may be specified as:-

$$\begin{aligned} d &= f + 2r \\ &= f + 2ux \\ &= 2ux + \sqrt{\frac{(x-2ux)^2 + (y-2ux)^2}{2\left[1+\left(\frac{y-2ux}{x-2ux}\right)^2\right]}.\left[1+\left(\frac{y-2ux}{x-2ux}\right)\right]^2} \end{aligned}$$

Equation 20

The First Numerical Solution Function, g(u)

To facilitate the employment of programs of numerical differentiation, and in particular The Ridders Method[3], it is convenient now to define a Solution function g(u). Note that this relates geometrically to d rather than f.

By re-arrangement of the simplified equation for d, Equation Twenty, we erect g(u) as:-

$$g(u) = d - 2ux - \sqrt{\frac{(x-2ux)^2 + (y-2ux)^2}{2\left[1+\left(\frac{y-2ux}{x-2ux}\right)^2\right]}.\left[1+\left(\frac{y-2ux}{x-2ux}\right)\right]^2}$$

Equation 21

The value of g(u) is zero when, in the presence of a calipered diagonal defined by measurement, the value of u specifies the relevant r: I.e. when g(u) is a root of the equation for d.

Therefore, a sufficient solution of Equation Twenty-One leads directly to mensuration of the castrum.

PART THREE
THE USE OF CALIPERED DIAGONAL d
TO ESTABLISH FILLET RADIUS r
USING THE RIDDERS METHOD

The method due to CJF Ridders is a late twentieth-century scheme for the iterative numerical differentiation of equations and the determination of roots thereby. It is very rapidly convergent for certain kinds of mathematical structures.

The Ridders Method needs to be presented with a continuous function, and three initial data that specify known point values.

The g(u) of Equation Twenty-One is algebraically-continuous, and in our context these starting values of u are convenient:-

$$u_1 = 0 \qquad \textbf{Equation 22a}$$

$$u_2 = 0.5 \qquad \textbf{Equation 22a}$$

$$u_3 = 0.25 \qquad \textbf{Equation 22a}$$

This is because zero and ½ are known extreme possibilities for u, whilst ¼ is their halfway position.

Effectively, our elaboration will always start at u=0.5 causing g(u) to diminish so that always $g(u_{n+1})<g(u_n)$.

Accordingly, the Ridders Criterion defined by:-

$$\operatorname{sgn}(xx, yy) = if\,((xx - yy) \geq 0,1,-1)$$

Condition 1

is always positive unity in our castrum mensuration situations.

The definition of The Ridders Method is:-

$$u_{n+3} = u_{n+2} + (u_{n+2} - u_n).\frac{(\operatorname{sgn}(g(u_n), g(u_{n+1}))).g(u_{n+2})}{\sqrt{g(u_{n+2})^2 - g(u_n).g(u_{n+1})}}$$

Equation 23

For which the first three iterates are clearly:-

$$u_4 = u_3 + (u_3 - u_1).\frac{(\operatorname{sgn}(g(u_1), g(u_2))).f(u_3)}{\sqrt{g(u_3)^2 - g(u_1).g(u_2)}} \qquad \textbf{Equation 24a}$$

$$u_5 = u_4 + (u_4 - u_2).\frac{(\operatorname{sgn}(g(u_2), g(u_3))).g(u_4)}{\sqrt{g(u_4)^2 - g(u_2).g(u_3)}} \qquad \textbf{Equation 24b}$$

$$u_6 = u_5 + (u_5 - u_3).\frac{(\operatorname{sgn}(g(u_3), g(u_4))).g(u_5)}{\sqrt{g(u_5)^2 - g(u_3).g(u_4)}} \qquad \textbf{Equation 24c}$$

These three points are potentially complex but usable values can be obtained as their complex moduli:-

$$\left| u_4 \right| = \sqrt{\Re(u_4)^2 + \Im(u_4)^2}$$ **Equation 25a**

$$\left| u_5 \right| = \sqrt{\Re(u_5)^2 + \Im(u_5)^2}$$ **Equation 25b**

$$\left| u_6 \right| = \sqrt{\Re(u_6)^2 + \Im(u_6)^2}$$ **Equation 25c**

Now when the first iterate u_4 is computed it is evident from defined-radius experiments with a variety of u and v predicates that the said u_4 is accurate to at least twelve places of decimals and that no further iteration shall improve that.

Therefore, we may justifiably adopt Equation Twenty-Four-a as a pseudoanalytic formula for castrum radius in terms of aspect ratio and calipered diagonal.

Since the sgn function is always +1 we may write:-

$$u_4 = \frac{1}{4} - \left(\frac{1}{4}\right) . \frac{g(u_3)}{\sqrt{g(u_3)^2 - g(u_1).g(u_2)}}$$

Equation 26

For notational convenience, the function $g_n \equiv g(u_n)$:-

$$u_4 = \frac{1}{4}\left[1 - \frac{g_3}{\sqrt{{g_3}^2 - g_1 g_2}}\right]$$

Equation 27

or:-

$$r = \frac{x}{4}\left[1 - \frac{g_3}{\sqrt{{g_3}^2 - g_1 g_2}}\right]$$

Equation 28

<u>Algebraic Renditions for the Functions g_1, g_2 and g_3</u>

To express g_1, g_2 and g_3 in computable forms we need to make relevant substitutions based upon the g(u) definitional Equation Twenty-One, as restated by Equation Twenty-Seven.

Simplification then gives convenient developments as:-

$$g_1 = d - \sqrt{\frac{x^2+y^2}{2\left[1+\left(\frac{y}{x}\right)^2\right]}.\left[1+\left(\frac{y}{x}\right)\right]^2}$$
$$= d - \frac{1}{2}\sqrt{2}x - \frac{1}{2}\sqrt{2}y$$
$$= d - \frac{\sqrt{2}}{2}(x+y)$$

Equation 29

g_2 (u = ½) presents a minor obstacle in that it is singular at u = 0.5 exactly. This is because the denominator a =x-2ux≡0, and therefore r=+∞.

This singularity is readily circumvented by defining an arbitrary infinitesimal m, and incorporating it into the expression for g_2.

Computationally, the value of m is immaterial, though I favor $+10^{-15}$ on the grounds that it is substantially below the required accuracy of u_4, assumed to be 10^{-12}.

By setting u=(1-m)/2, a statement of g_2 may thus be given as:-

$$g_2 = d - (1-m)x - \sqrt{\frac{(x-(1-m)x)^2 + (y-(1-m)x)^2}{2\left[1+\left(\frac{y-(1-m)x}{x-(1-m)x}\right)^2\right]}.\left(1+\left(\frac{y-(1-m)x}{x-(1-m)x}\right)\right)^2}$$

Equation 30

or:-

$$g_2 = d - x + xm - \sqrt{2}xm - \frac{1}{2}\sqrt{2}y + \frac{1}{2}\sqrt{2}x$$
$$= d - x + mx\left(1-\sqrt{2}\right) - \frac{\sqrt{2}}{2}(y-x)$$
$$= d - x - \frac{\sqrt{2}}{2}(y-x)$$

Equation 31

and:-

$$g_3 = d - \frac{x}{2} - \sqrt{\frac{\left(\frac{x}{2}\right)^2 + \left(y - \frac{x}{2}\right)^2}{2\left[1 + \left(\frac{y - \frac{x}{2}}{\frac{x}{2}}\right)^2\right]} . \left[1 + \left(\frac{y - \frac{x}{2}}{\frac{x}{2}}\right)\right]^2}$$

$$= d - \frac{1}{2}x - \frac{1}{2}\sqrt{2}y$$

$$= d - \frac{1}{2}\left(x + \sqrt{2}y\right)$$

Equation 32

Furthermore:-

$$g_3{}^2 = \left[d - \frac{1}{2}\left(x + \sqrt{2}y\right)\right].\left[d - \frac{1}{2}\left(x + \sqrt{2}y\right)\right]$$

$$= \frac{1}{4}\left(-2d + x + \sqrt{2}y\right)^2$$

$$= \frac{1}{4}\left(x + \sqrt{2}y - 2d\right)^2$$

Equation 33

and:-

$$g_1 g_2 = \left[d - \frac{\sqrt{2}}{2}(x + y)\right].\left[d - x - \frac{\sqrt{2}}{2}(y - x)\right]$$

$$= \frac{-1}{4}\left(-2d + \sqrt{2}x + \sqrt{2}y\right)\left(2d - 2x - \sqrt{2}y + \sqrt{2}x\right)$$

$$= \left[d - \frac{\sqrt{2}}{2}(x + y)\right].\left[2(d - x) - \sqrt{2}(y - x)\right]$$

Equation 34

At this juncture it is convenient to resolve the square-rooted denominator featured in Equation Twenty-Seven. We may note that:-

$$g_3{}^2 - g_1 g_2 = \frac{1}{4}\left(x + \sqrt{2}y - 2d\right)^2 - \left[d - \frac{\sqrt{2}}{2}(x + y)\right].\left[2(d - x) - \sqrt{2}(y - x)\right]$$

$$= \frac{-1}{4}x^2\left(-3 + 2\sqrt{2}\right)$$

$$= \frac{x^2}{4}\left(3 - 2\sqrt{2}\right)$$

Equation 35

So that:-

$$\sqrt{{g_3}^2 - g_1 g_2} = \frac{x}{2}\sqrt{3-2\sqrt{2}}$$

Equation 36

The appropriate substitutions in Equation Twenty-Seven then give:-

$$u_4 = \frac{1}{4}\left[1 - \frac{d - \frac{1}{2}\left(x+\sqrt{2}y\right)}{\frac{x}{2}\sqrt{3-2\sqrt{2}}}\right]$$

Equation 37

$$r = \frac{x}{4}\left[1 - \frac{d - \frac{1}{2}\left(x+\sqrt{2}y\right)}{\frac{x}{2}\sqrt{3-2\sqrt{2}}}\right]$$

Equation 38

PART FOUR
SIMPLIFICATION OF THE RIDDERS EXPRESSION FOR FILLET RADIUS

Equation Thirty-Eight is vastly simpler and cheaper than the equivalent expression of my prior pseudoanalysis presented in “Pseudoanalytic Mensuration of the Castrum by Means of The Ridders Method”.

Nevertheless, it is still not in its simplest possible form.

The following development will help us to identify a formula for corner radius that implicates x, y, d, constants and a few cardinal operators only: Seemingly a tall order but surprisingly possible.

The first phase of development is elaborated in Equation Thirty-Nine:-

.

$$\begin{aligned}
u_4 &= \frac{1}{4}\left[1 - \frac{d - \frac{1}{2}\left(x+\sqrt{2}y\right)}{\frac{x}{2}\sqrt{3-2\sqrt{2}}}\right] \\
&= \frac{1}{4}\left[1 - \frac{2d - x - \sqrt{2}y}{x\sqrt{3-2\sqrt{2}}}\right] \\
&= \frac{1}{4}\left[1 - \frac{2d}{x\sqrt{3-2\sqrt{2}}} + \frac{x}{x\sqrt{3-2\sqrt{2}}} + \frac{\sqrt{2}y}{x\sqrt{3-2\sqrt{2}}}\right] \\
&= \frac{1}{4}\left[1 - \frac{\left(2+\left(\sqrt{2}\right)^3\right)d}{x} + \frac{1}{\sqrt{3-2\sqrt{2}}} + \left(2+\sqrt{2}\right)v\right]
\end{aligned}$$

Equation 39

Condensation in terms of the isolation of 2+n2^½ may then be continued as:-

$$\begin{aligned}
u_4 &= \frac{1}{4}\left[1+\frac{1}{\sqrt{3-2\sqrt{2}}}+\left(2+\sqrt{2}\right)v-\frac{\left(2+\sqrt{2}^3\right)d}{x}\right] \\
&= \frac{1}{4}\left[1+\frac{1}{\sqrt{3-2\sqrt{2}}}+\left(2+\sqrt{2}\right)v-\frac{\left(2+2\sqrt{2}\right)d}{x}\right] \\
&= \frac{1}{4}\left[1+\frac{1}{\sqrt{3-2\sqrt{2}}}+\left(2+\sqrt{2}\right)v-\frac{\sqrt{2}\left(2+\sqrt{2}\right)d}{x}\right] \\
&= \frac{1}{4}\left[1+\frac{1}{\sqrt{3-2\sqrt{2}}}+\left(2+\sqrt{2}\right)\left(v-\frac{\sqrt{2}d}{x}\right)\right] \\
&= \frac{1}{4}\left[1+\frac{1}{\sqrt{3-2\sqrt{2}}}+\left(2+\sqrt{2}\right)\left(\frac{y}{x}-\frac{\sqrt{2}d}{x}\right)\right] \\
&= \frac{1}{4}\left[1+\frac{1}{\sqrt{3-2\sqrt{2}}}+\left(2+\sqrt{2}\right)\left(\frac{y-\sqrt{2}d}{x}\right)\right] \\
&= \frac{1}{4}\left[1+\frac{1}{\sqrt{3-2\sqrt{2}}}\right]+\frac{2+\sqrt{2}}{4}.\frac{y-\sqrt{2}d}{x}
\end{aligned}$$

Equation 40

At this stage is appropriate to note the following system of surd identities:-

$$\frac{1}{4}\left(1+\frac{1}{\sqrt{3-2\sqrt{2}}}\right)=\frac{1}{4}\left(1+\frac{1}{\sqrt{2}-1}\right)=\frac{1}{4}\left(1+1+\sqrt{2}\right)=\frac{2+\sqrt{2}}{4}$$

Equation 41

Accordingly, u_4 becomes:-

$$\begin{aligned}
u_4 &= \frac{2+\sqrt{2}}{4}+\frac{2+\sqrt{2}}{4}.\frac{y-\sqrt{2}d}{x} \\
&= \frac{2+\sqrt{2}}{4}.\left(1+\frac{y-\sqrt{2}d}{x}\right)
\end{aligned}$$

Equation 42

Therefore it is clear that the Corner Radius, r, is given by:-

$$r=\frac{2+\sqrt{2}}{4}.\left(1+\frac{y-\sqrt{2}d}{x}\right).x$$

Equation 43

or:-

$$\boxed{r=\frac{2+\sqrt{2}}{4}.\left(x+y-\sqrt{2}d\right)}$$

Equation 44

By evaluation of the constants this yields the following numerical equation good for twelve-figure accuracy:-

$$r=0.853553390593274.\left(x+y-1.414213562373095d\right)$$

Equation 45

PART FIVE
DISCUSSION OF THE METHOD

Equation Forty-Four offers two Square Roots; two each of multiplications and additions; and one each of divisions and subtractions, in contrast to the numerous operations of the trigonometric equivalent described at the outset.

If you are prepared to work with the numerical estimator Equation Forty-Five, then there are merely two each of multiplications, additions and subtractions, and we have achieved our goal of a corner radius formula of cardinal operators exclusively.

For the six test data specified in the Appendices, The Ridders Method determine u_4 to at least fifteen decimal places, the limit of accuracy for Mathcad Student Edition 2.0, and Corner Radius is computed to fourteen figures.

For the test data of Figure One u_5 is also 0.125 and hence precise to at least fifteen places. u_6 and u_7 are however markedly deteriorated, accurate to no better than seven figures. Clearly, therefore, The Ridders Method must not needlessly be pressed.

For test purposes, Prior and Posterior Area and Perimeter values were computed. These were respectively identical to the limits of accuracy.

Excel® tabulations for the Measured d Test Data are included in Appendix B.

When I was young, scientists and engineers in British institutions had the vague habit of calling precise relationships “quantitative”.

The unstated implication was that they thought the determined correlation the outcome of some exact natural law, but that they were unable or unwilling to define it.

Our pseudoanalysis generates numerical outcomes so nearly exact that as practitioners we consider the formula good enough.

Accordingly, we congratulate ourselves that it is a significant contribution, if not to knowledge, then at least to Goodenough Theory, and swell with pride in our offering at the altar of the national art form.

Needless to add, pure mathematicians would always object that any equation that cannot continuously be traced from a set of axioms must perforce be defective. But we answer that it is human understanding that is defective and provisional, if even the Postulates of Euclid cannot be trusted.

References

1 http://en.wikipedia.org/wiki/List_of_trigonometric_identities

2 http://en.wikipedia.org/wiki/Inverse_trigonometric_functions

3 "A New Algorithm for Computing a Single Root of a Real Continuous Function"
CJF Ridders
IEEE Transactions on Circuits and Systems
Vol. CAS-26. No. 11 November 1979
pp 979-980

APPENDIX A
CASTRUM DIAGAMS
FOR TRIAL COMPUTATIONS

Figure Number	Grid Dimensions in Unit Squares		Grid Dimensions in Millimeters		Mean Square Side Length in Millimeters	Figure Diagonals in Millimeters			Prior Metrics			Figure Diagonals in Centimeters		
	H	V	Width	Height		d	f	h	r	x	y	d	f	h
One	20	20	151.2	151.8	7.575	111.4	95.9	101.5	1	8	14	14.713555	12.666336	13.405977
Two	20	20	151.3	151.8	7.5775	107	80.1	86.3	1.773	8	14	14.132409	10.579495	11.398382
Three	20	20	151	151.6	7.565	99.1	53.9	62.4	3	8	14	13.088988	7.119036	8.241704
Four	20	20	151.1	151.6	7.5675	77.9	32.3	34.1	3	8	10	10.288922	4.266138	4.503880
μ					7.57125									
σ					0.00515									
CV%					0.06807									

Table A1
Source Data for Castrum Geometry

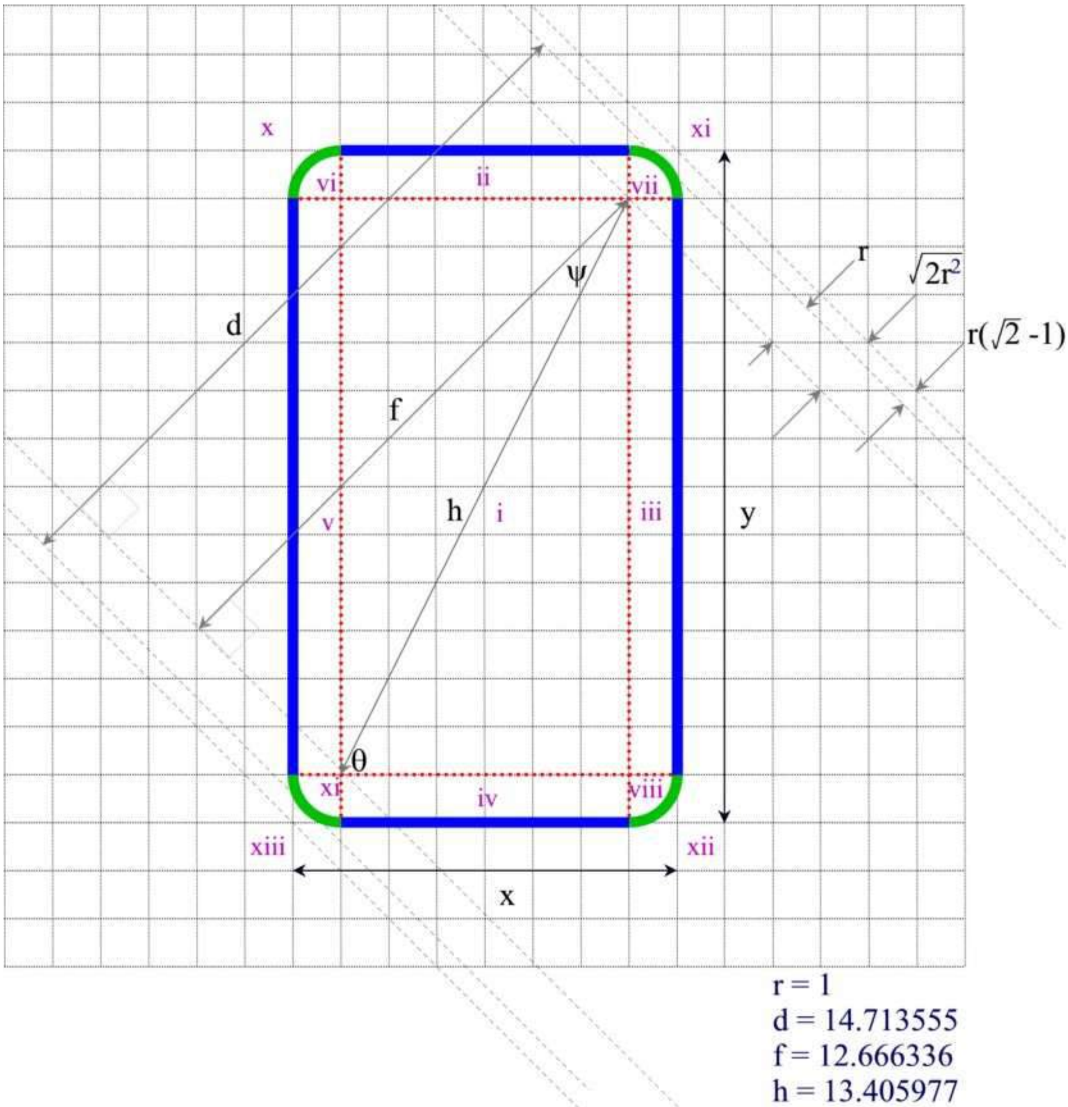

Figure A1
Castrum Geometry for
r = 1; x = 8; y = 14
d, f and h are as measured from paper

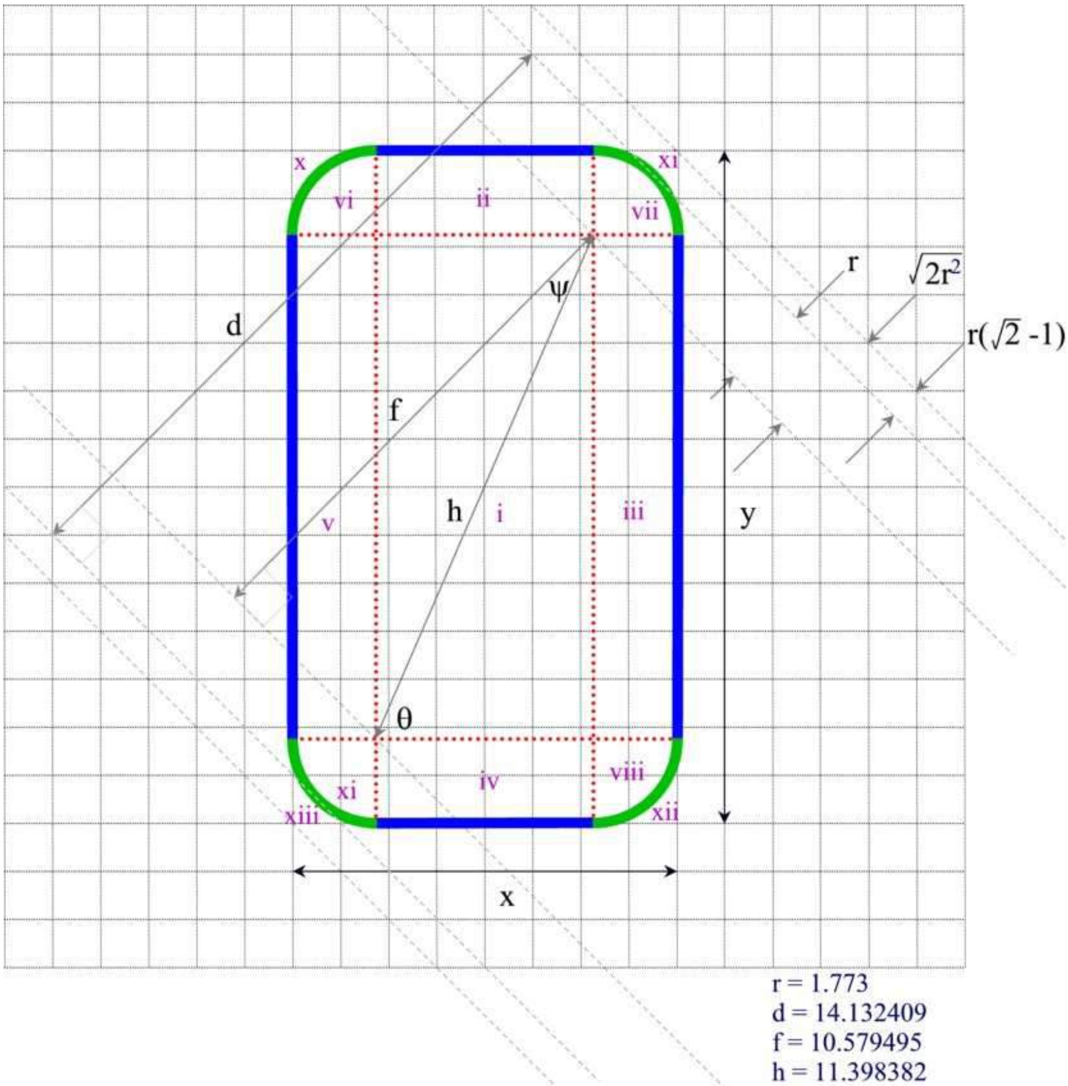

Figure A2
Castrum Geometry for
r = 1.773; x = 8; y = 14
d, f and h are as measured from paper

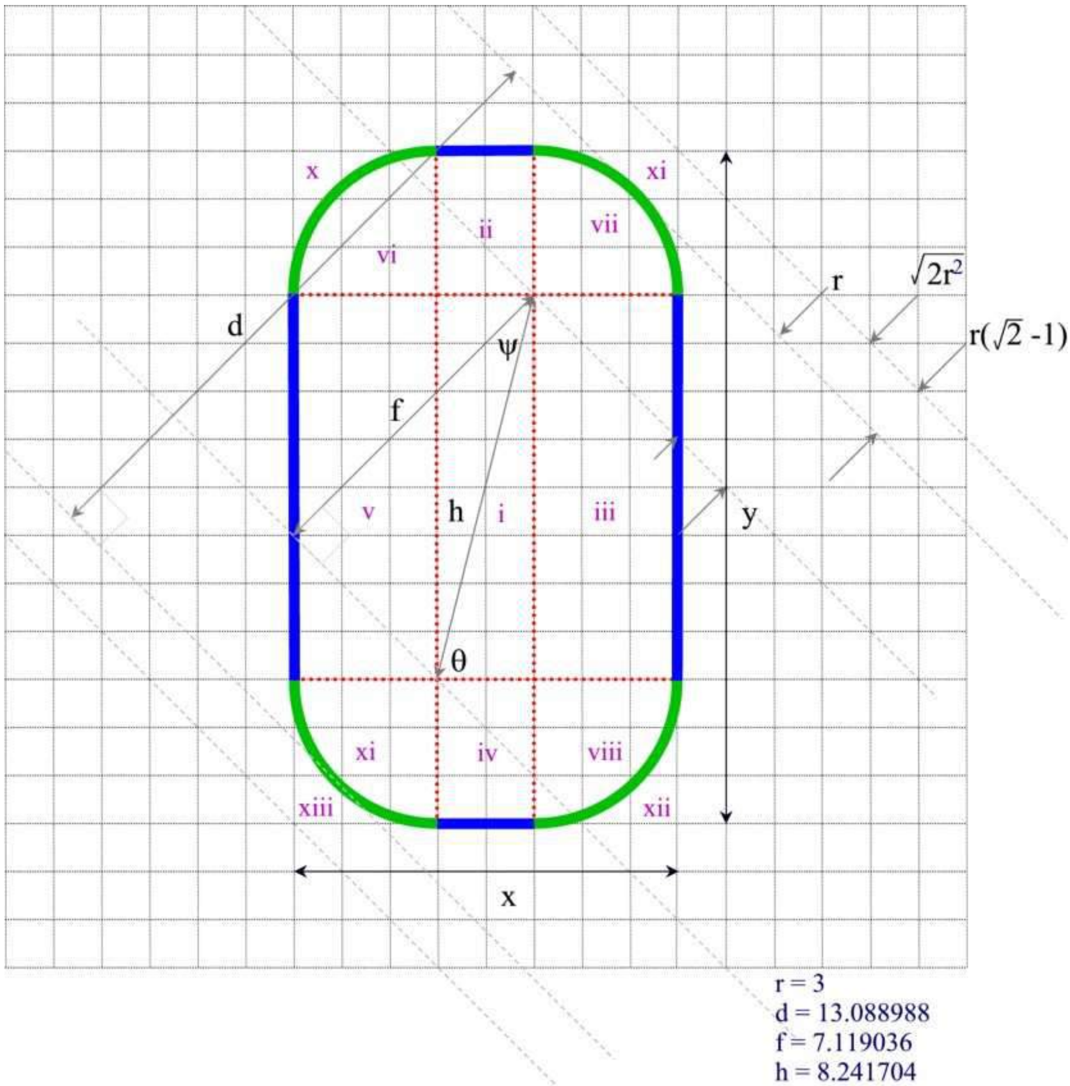

Figure A3
Castrum Geometry for
r = 3; x = 8; y = 14
d, f and h are as measured from paper

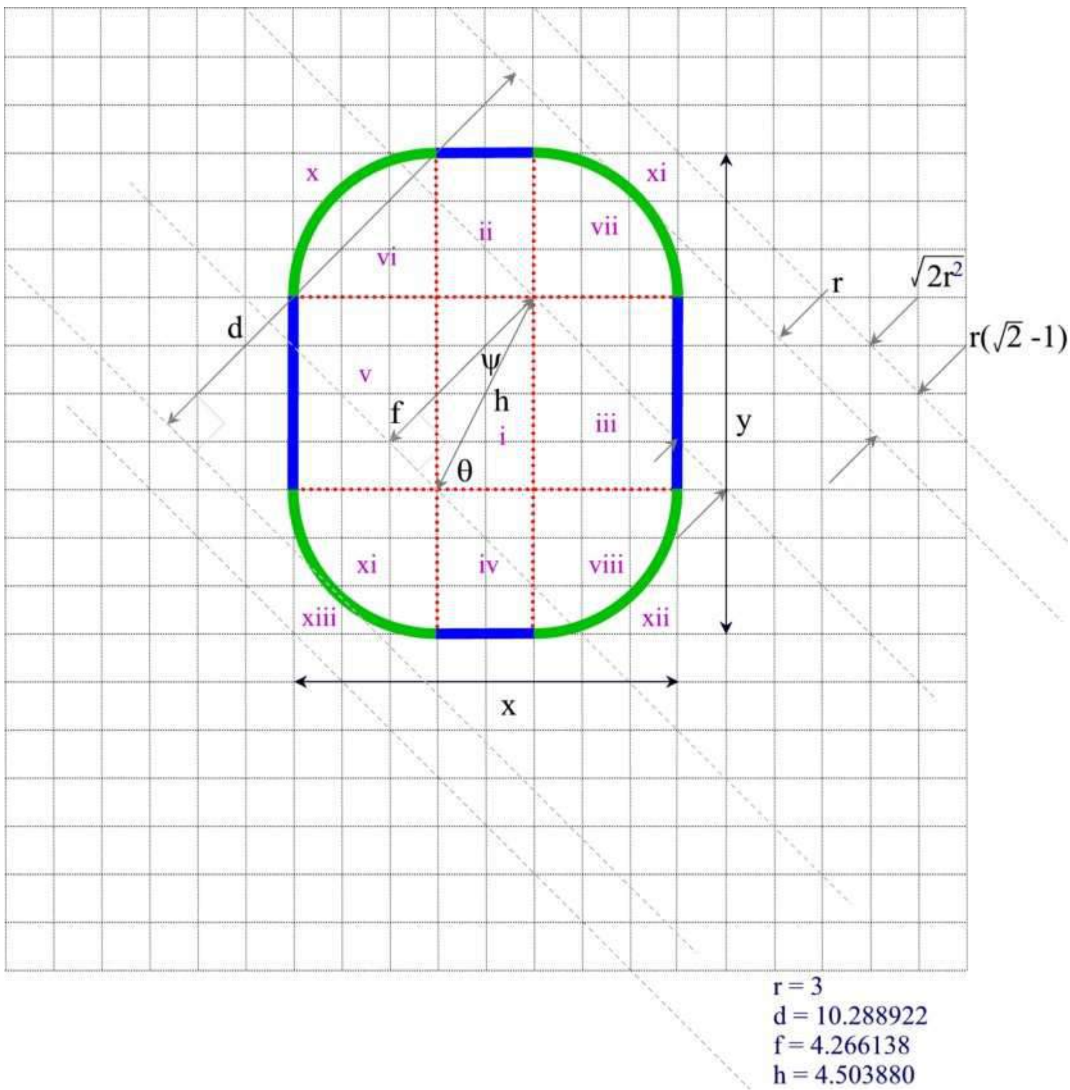

Figure A4
Castrum Geometry for
r = 3; x = 8; y = 10
d, f and h are as measured from paper

APPENDIX B
CASTRUM EXCEL® WORKSHEETS
FOR TRIAL COMPUTATIONS

NAME:	CASTRUM Figure One	
Datum	**Symbol**	**Value**
Width	x	8
Height	y	14
Measured Corner Radius	r_m	1
Measured Calipered Diagonal	d_m	14.713555
Zone i Width	a	6
Zone i Height	b	12
Zone i Aspect Ratio	ρ	2
Zone i Diagonal	h	13.41640786
Dimensionless Radius	u	0.125
Zone i Base Angle	θ	1.107148718
Zone i Calipered Height Angle	ψ	0.321750554
Zone i Calipered Diagonal	f	12.72792206
Castrum Calipered Diagonal	d	14.72792206
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Ridder Root (Actual Dimensionless Radius)	u_4	0.127167822
Actual Corner Radius according to Ridder's Method	r_r	1.017342577
Prior Castrum Area	A_{pr}	111.1415927
Posterior Castrum Area	A_{po}	111.1115605
Prior Castrum Perimeter	P_{pr}	42.28318531
Posterior Castrum Perimeter	P_{po}	42.25341132
Area Specific Defect (Percentage)	SD_A	-0.02702154
Perimeter Specific Defect (Percentage)	SD_P	-0.070415677

Table B1
Castrum Trial Data and Results for
Figure One

NAME:	CASTRUM Figure Two	
Datum	**Symbol**	**Value**
Width	x	8
Height	y	14
Measured Corner Radius	r_m	1.773
Measured Calipered Diagonal	d_m	14.132409
Zone i Width	a	4.454
Zone i Height	b	10.454
Zone i Aspect Ratio	ρ	2.347103727
Zone i Diagonal	h	11.36328438
Dimensionless Radius	u	0.221625
Zone i Base Angle	θ	1.168030722
Zone i Calipered Height Angle	ψ	0.382632558
Zone i Calipered Diagonal	f	10.54154789
Castrum Calipered Diagonal	d	14.08754789
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Ridder Root (Actual Dimensionless Radius)	u_4	0.214855982
Actual Corner Radius according to Ridder's Method	r_r	1.718847855
Prior Castrum Area	A_{pr}	109.3015716
Posterior Castrum Area	A_{po}	109.4638888
Prior Castrum Perimeter	P_{pr}	40.95608755
Posterior Castrum Perimeter	P_{po}	41.04905675
Area Specific Defect (Percentage)	SD_A	0.148503948
Perimeter Specific Defect (Percentage)	SD_P	0.226997265

Table B2
Castrum Trial Data and Results for
Figure Two

NAME:	CASTRUM Figure Three	
Datum	**Symbol**	**Value**
Width	x	8
Height	y	14
Measured Corner Radius	r_m	3
Measured Calipered Diagonal	d_m	13.088988
Zone i Width	a	2
Zone i Height	b	8
Zone i Aspect Ratio	ρ	4
Zone i Diagonal	h	8.246211251
Dimensionless Radius	u	0.375
Zone i Base Angle	θ	1.325817664
Zone i Calipered Height Angle	ψ	0.5404195
Zone i Calipered Diagonal	f	7.071067812
Castrum Calipered Diagonal	d	13.07106781
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Ridder Root (Actual Dimensionless Radius)	u_4	0.372296052
Actual Corner Radius according to Ridder's Method	r_r	2.978368419
Prior Castrum Area	A_{pr}	104.2743339
Posterior Castrum Area	A_{po}	104.3853445
Prior Castrum Perimeter	P_{pr}	38.84955592
Posterior Castrum Perimeter	P_{po}	38.88669334
Area Specific Defect (Percentage)	SD_A	0.106460115
Perimeter Specific Defect (Percentage)	SD_P	0.095592896

Table B3
Castrum Trial Data and Results for
Figure Three

NAME:	CASTRUM Figure Four	
Datum	**Symbol**	**Value**
Width	x	8
Height	y	10
Measured Corner Radius	r_m	3
Measured Calipered Diagonal	d_m	10.288922
Zone i Width	a	2
Zone i Height	b	4
Zone i Aspect Ratio	ρ	2
Zone i Diagonal	h	4.472135955
Dimensionless Radius	u	0.375
Zone i Base Angle	θ	1.107148718
Zone i Calipered Height Angle	ψ	0.321750554
Zone i Calipered Diagonal	f	4.242640687
Castrum Calipered Diagonal	d	10.24264069
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Ridder Root (Actual Dimensionless Radius)	u_4	0.368016689
Actual Corner Radius according to Ridder's Method	r_r	2.944133513
Prior Castrum Area	A_{pr}	72.27433388
Posterior Castrum Area	A_{po}	72.55939195
Prior Castrum Perimeter	P_{pr}	30.84955592
Posterior Castrum Perimeter	P_{po}	30.94546833
Area Specific Defect (Percentage)	SD_A	0.394411204
Perimeter Specific Defect (Percentage)	SD_P	0.310903681

Table B4
Castrum Trial Data and Results for
Figure Four

NAME:	CASTRUM Unit Circle (with error control)	
Datum	**Symbol**	**Value**
Width	x	1
Height	y	1
Measured Corner Radius	r_m	0.5
Measured Calipered Diagonal	d_m	14.72990777
Error Control Infinitesimal	ε	1E-15
Zone i Width	a	1E-15
Zone i Height	b	0
Zone i Aspect Ratio	ρ	0
Zone i Diagonal	h	0
Dimensionless Radius	u	0.5
Zone i Base Angle	θ	#DIV/0!
Zone i Calipered Height Angle	ψ	#DIV/0!
Zone i Calipered Diagonal	f	7.07107E-16
Castrum Calipered Diagonal	d	1
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Ridder Root (Actual Dimensionless Radius)	u_4	0.5
Actual Corner Radius according to Ridder's Method	r_r	0.5
Prior Castrum Area	A_{pr}	0.785398163
Posterior Castrum Area	A_{po}	0.785398163
Prior Castrum Perimeter	P_{pr}	3.141592654
Posterior Castrum Perimeter	P_{po}	3.141592654
Area Specific Defect (Percentage)	SD_A	8.48148E-14
Perimeter Specific Defect (Percentage)	SD_P	4.24074E-14

Table B5
Castrum Trial Data and Results for
The Unit Circle

NAME:	CASTRUM Unit Square	
Datum	**Symbol**	**Value**
Width	x	1
Height	y	1
Measured Corner Radius	r_m	0
Measured Calipered Diagonal	d_m	14.72990777
Zone i Width	a	1
Zone i Height	b	1
Zone i Aspect Ratio	ρ	1
Zone i Diagonal	h	1.414213562
Dimensionless Radius	u	0
Zone i Base Angle	θ	0.785398163
Zone i Calipered Height Angle	ψ	0
Zone i Calipered Diagonal	f	1.414213562
Castrum Calipered Diagonal	d	1.414213562
Lower Bound Functional u	u_1	0
Upper Bound Functional u	u_2	0.5
Proximative Functional u	u_3	0.25
First Ridder Root (Actual Dimensionless Radius)	u_4	-3.79054E-16
Actual Corner Radius according to Ridder's Method	r_r	-3.79054E-16
Prior Castrum Area	A_{pr}	1
Posterior Castrum Area	A_{po}	1
Prior Castrum Perimeter	P_{pr}	4
Posterior Castrum Perimeter	P_{po}	4
Area Specific Defect (Percentage)	SD_A	0
Perimeter Specific Defect (Percentage)	SD_P	2.22045E-14

Table B6
Castrum Trial Data and Results for
The Unit Square