A Simple but Accurate Approximator of The Complete Elliptic Integral of the First Kind

by
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PART I
METHODOLOGICAL REVIEW

The Complete Elliptic Integral of the First Kind, K(k), is defined in Abramowitz and Stegun[1] as:-

$$K(k) = CEIF(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - x.\sin(\theta)^2}}.d\theta = \int_0^1 \frac{1}{\sqrt{(1-t^2).(1-x.t^2)}}.dt$$

Equation One

There is no perfect closed-form expression for the exact CEIF(k).

There are numerous schemes of numerical integration and many closed-form approximators that are more or less accurate estimators of The Complete Elliptic Integral of the First Kind.

There are also "general" numerical approaches such as Rule 18 that notably fail to give usable values.

<u>Standard Approximators</u>

Double Factorial Quotient Series

This may be summarised as:-

$$K_{series}(k) = \frac{1}{2}\pi\left[1 - \sum_{i=1}^{n} k^i \left(\frac{\prod_{j=1}^{i} 2j - 1}{\prod_{j=1}^{i} 2j}\right)^2\right]$$

Equation Two

When n = 15 and k = ½ the accuracy is to five mantissa digits.

Simpson's Rule

Allow that:-

$$g_0 = \frac{1}{\sqrt{1 - k.\sin(0)^2}} = 1$$

Equation Three a

$$g_n = \frac{1}{\sqrt{1 - k.\sin\left(\frac{\pi}{2}\right)^2}} = 1.41421356237 \equiv \sqrt{2}$$

Equation Three b

$$g_i = \frac{1}{\sqrt{1 - k.\sin\left(\frac{i}{n}.\frac{\pi}{2}\right)^2}}$$

Equation Three c

Then:-

$$K_{simpson}(k) \approx \frac{h}{3}\left[g_0 + g_n + \sum_{i=1}^{n-1}\left[3 - (-1)^i\right].g_i\right]$$

Equation Four

where:-

$$h = \frac{\pi}{2n}$$

Equation Five

Simpson's Rule[2] works on the basis that there are an odd number of data values (i.e. an even number of intervals). Simpson's Rule is one of the simplest and most-elegantly programmable, as well as one of the most robust, of the family of Closed Newton-Cotes numerical integration formulae.

It is a classical scheme good for many continuous functions whose trajectories can realistically be modelled as a succession of parabolas.

When n = 48 (giving 49 data inclusive of the zeroth) and k = ½, the accuracy of $K_{simpson}(k)$ is nine mantissa digits.

The Hastings Polynomial Approximation

Hastings[3] provides a high-precision double polynomial for estimating values of CEIF(k):-

$$K_{hastings}(k) = \sum_{i=0}^{4} a_i \, . k^i + (\log_n k) \sum_{i=0}^{4} b_i \, . k^i$$

Equation Six

Table One presents the requisite coefficients of the relevant Hastings Polynomials:-

a_0	1.38629436112	b_0	0.50000000000
a_1	0.09666344259	b_1	0.12498593597
a_2	0.03590092383	b_2	0.06880248576
a_3	0.03742563713	b_3	0.03328355346
a_4	0.01451196212	b_4	0.00441787012

Table One
Hastings Polynomial Coefficients for CEIF Estimation

For k = ½ the Hastings Polynomial system yields nine-digit accuracy.

PART II
APPROXIMATIONS DEVELOPED FROM THE ARITHMETIC-GEOMETRIC MEAN

A definition of The Complete Elliptic Integral of the First Kind in terms of The Arithmetic-Geometric Mean is furnished by the Gauss-Lagrange Formula[4]:-

$$K(k) = CEIF(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - x.\sin(\alpha)^2}}.d\theta = \frac{\pi}{2.AGM(1, \sin(\alpha))}$$

Equation Seven

which may conveniently be tabulated as:-

$$K(k) = CEIF(k) = \int_0^{1} \frac{1}{\sqrt{(1 - t^2).(1 - k.t^2)}}.dt = \frac{\pi}{2.AGM(1, \sqrt{k})}$$

Equation Eight

where:-

$$\alpha = \sin^{-1}\left(\sqrt{k}\right)$$

Equation Nine

In the context of Equation Eight K is the Complete Elliptic Integral of the First Kind for Argument k where 0<k≤1, and we shall develop AGM processes on this interval.

π is the Ludolphine Constant and AGM(x,y) is the Arithmetic-Geometric Mean of x and y defined by:-

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equation Ten a

$$\begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} = \begin{pmatrix} \frac{a_i + b_i}{2} \\ \sqrt{a_i b_i} \end{pmatrix}$$

Equation Ten b

$$AGM(x, y) = \begin{pmatrix} \frac{x + y}{2} \\ \sqrt{xy} \end{pmatrix}$$

Equation Ten c

whilst:-

$$c_i = \frac{a_i + b_i}{2}$$

Equation Ten d

This iteration will yield fifteen-figure accuracy in four passes when presented with the initial conditions:-

$$a_0 = 1$$

Equation Eleven a

$$b_0 = \sin(\alpha)$$

Equation Eleven b

$$c_0 = \cos(\alpha)$$

Equation Eleven c

Approximation of the AGM

A three-pass AGM can achieve a precision sufficiently high for many purposes and I dare say for all environmental science applications.

By internal substitutions it is possible to express a three-pass AGM by the construct:-

$$AGM_{(2)} = \frac{\dfrac{\dfrac{\dfrac{x+y}{2}+\sqrt{xy}}{2}+\sqrt{\dfrac{x+y}{2}.\sqrt{xy}}}{2}+\sqrt{\dfrac{\dfrac{x+y}{2}+\sqrt{xy}}{2}+\sqrt{\dfrac{x+y}{2}.\sqrt{xy}}}}{2}$$

Equation Twelve

A degree of simplification gives:-

$$AGM_{(2)} = \frac{x}{2^4}+\frac{y}{2^4}+\frac{1}{2^3}\left(x^{\frac{1}{2}}y^{\frac{1}{2}}\right)+\frac{1}{2^{\frac{5}{3}}}\sqrt{x+y}.x^{\frac{1}{4}}y^{\frac{1}{4}}+\frac{1}{2^{\frac{9}{4}}}\sqrt{x+y+2\sqrt{x}\sqrt{y}}.(x+y)^{\frac{1}{4}}.x^{\frac{1}{8}}y^{\frac{1}{8}}$$

Equation Thirteen

and further simplifications bring us to:-

$$AGM_{(2)} = 2^{-\frac{5}{4}}\sqrt{x+y+2\sqrt{xy}}.(x+y)^{\frac{1}{4}}.(xy)^{\frac{1}{8}}$$

Equation Fourteen

For a system where $a_0 = 1$ and $b_0 = \sin(\alpha)$ we may note the identities:-

$$AGM_{(2)} = \frac{\left(x+y+2\sqrt{xy}\right)^{\frac{1}{2}}(x+y)^{\frac{1}{4}}.(xy)^{\frac{1}{8}}}{\left(\sqrt{32}\right)^{\frac{1}{2}}}$$

$$= \frac{\left(1+\sin(\alpha)+2\sqrt{\sin(\alpha)}\right)^{\frac{1}{2}}(1+\sin(\alpha))^{\frac{1}{4}}.(\sin(\alpha))^{\frac{1}{8}}}{\left(\sqrt{32}\right)^{\frac{1}{2}}}$$

$$= 2^{-\frac{5}{4}}(1+\sin(\alpha))\sqrt{\sqrt{1+\sin(\alpha)}.\sqrt[4]{\sin(\alpha)}}$$

Equation Fifteen

Any of these identities reduce to:-

$$AGM_{(2)} = \frac{(1+\sin(\alpha))^{\frac{1}{4}}.\left[(\sin(\alpha))^{\frac{1}{8}} + (\sin(\alpha))^{\frac{5}{8}}\right]}{\sqrt[4]{32}}$$

Equation Sixteen

The Specific Defect, $SpDef_{AGM\text{-}PA}$, is given by:-

$$SpDef_{AGM-Eqn.16} = 100\frac{Eqn.16 - AGM}{AGM}$$

Equation Seventeen

For the case of:-

$$\sin(\alpha) = \sqrt{\frac{1}{2}} = 0.707106781186548 \equiv \cos(\alpha)$$

the Specific Defect vis-a-vis the full AGM is -0.000000000048646, or 12-figure accuracy for the pseudoanalytic form.

Logarithmic Development of the Approximation

As adumbrated above, the resolution of Equation Twelve in terms of logarithms yields the simplification:-

$$AGM_{(2)} = \frac{\left(x+y+2\sqrt{xy}\right)^{\frac{1}{2}}(x+y)^{\frac{1}{4}}.(xy)^{\frac{1}{8}}}{\sqrt[4]{32}}$$

Equation Fifteen a

Appropriate substitutions of x = 1 and y = sin(α) then give:-

$$AGM_{(2)} = \frac{\left(1+\sin(\alpha)+2\sqrt{\sin(\alpha}\right)^{\frac{1}{2}}(1+\sin(\alpha))^{\frac{1}{4}}.(\sin(\alpha))^{\frac{1}{8}}}{\sqrt[4]{32}}$$

Equation Fifteen b

whilst further simplification of the above expression brings us:-

$$AGM_{(2)} = 2^{-\frac{5}{4}}\left(1+\sin(\alpha)+2\sqrt{\sin(\alpha}\right)^{\frac{1}{2}}(1+\sin(\alpha))^{\frac{1}{4}}.(\sin(\alpha))^{\frac{1}{8}}$$

Equation Fifteen c

Continued simplification of the above expression brings us to:-

$$AGM_{(2)} = 2^{-\frac{5}{4}}\left[\left(1+\sqrt{\sin(\alpha)}\right).(1+\sin(\alpha))^{\frac{1}{4}}.(\sin(\alpha))^{\frac{1}{8}}\right]$$

Equation Sixteen

Whereupon substitution of $\sqrt{k}$ for sin(α) gives:-

$$AGM_{(2)} = 2^{-\frac{5}{4}}.\left(1+k^{\frac{1}{4}}\right)\left(1+\sqrt{k}\right)^{\frac{1}{4}} k^{\frac{1}{16}}$$

Equation Eighteen

Trigonometric Development of the Approximation

It can also be demonstrated (as I have elsewhere) that a trigonometrical resolution of Equation Twelve yields the simplification:-

$$AGM_{(2)} = \sqrt{2^{-\frac{5}{2}}.\left[\left(\frac{v}{\sin(\theta)}\right)+2\left(\frac{v}{2}\right)\right].\left(\frac{v}{\sin(\theta)}\right)^{\frac{1}{4}}}.\left(\sqrt{xy}\right)^{\frac{1}{4}}$$

Equation Nineteen

where as before x = 1 and y = sin(α).
v is defined by:-

$$\begin{aligned} v &= \sin(\alpha).(x+y) \\ &= \sqrt{(x+y)^2-(x-y)^2} \\ &= \sqrt{(1+\sin(\alpha))^2-(1-\sin(\alpha))^2} \\ &= 2\sqrt{\sin(\alpha)} \end{aligned}$$

Equation Twenty

whilst:-

$$\theta = \cos^{-1}\left(\frac{x-y}{x+y}\right)$$

Equation Twenty-One

so that:-

$$\sin(\theta) = \sin\left[\cos^{-1}\left(\frac{x-y}{x+y}\right)\right] = \sin\left[\cos^{-1}\left(\frac{1-\sin(\alpha)}{1+\sin(\alpha)}\right)\right] = \frac{2\sqrt{\sin(\alpha)}}{1+\sin(\alpha)}$$

Equation Twenty-Two

Substitution then permits us to re-write Equation Nineteen as:-

$$AGM_{(2)} = \sqrt{2^{-2.5}.\left[\left(\frac{2\sqrt{\sin(\alpha)}}{\frac{2\sqrt{\sin(\alpha)}}{1+\sin(\alpha)}}\right)+2\left(\frac{2\sqrt{\sin(\alpha)}}{2}\right)\right]}.\left(\frac{2\sqrt{\sin(\alpha)}}{\frac{2\sqrt{\sin(\alpha)}}{1+\sin(\alpha)}}\right)^{\frac{1}{4}}.\left(\sqrt{\sin(\alpha)}\right)^{\frac{1}{4}}$$

Equation Twenty-Three

Appropriate cancellations yield:-

$$AGM_{(2)} = \sqrt{2^{-2.5}.\left[(1+\sin(\alpha))+2\sqrt{\sin(\alpha)}\right]}.(1+\sin(\alpha))^{\frac{1}{4}}.\left(\sqrt{\sin(\alpha)}\right)^{\frac{1}{4}}$$

Equation Twenty-Four

which reduces to:-

$$AGM_{(2)} = 2^{-\frac{5}{4}}.\left(1+\sqrt{\sin(\alpha)}\right).(1+\sin(\alpha))^{\frac{1}{4}}.\left(\sqrt{\sin(\alpha)}\right)^{\frac{1}{4}}$$

Equation Twenty-Five

The substitution $\sqrt{k}$ = sin(α) then gives:-

$$AGM_{(2)} = 2^{-\frac{5}{4}}.\left(1+k^{\frac{1}{4}}\right).\left(1+\sqrt{k}\right)^{\frac{1}{4}}.k^{\frac{1}{16}}$$

Equation Twenty-Six

Equations Eighteen and Twenty-Six are of course identical.

Either of them gives a specific defect of –0.00000000486462 relative to the full AGM, that is to say ten-figure accuracy.

The Use of Closed-Form AGM Approximations

By combining Equations Seven and Eight the Complete Elliptic Integral of the First Kind, K(k), may further be defined as:-

$$K(k) = \frac{1}{2.AGM(1,\sin(\alpha))} = \frac{1}{2.AGM\left(1,\sqrt{k}\right)}$$

Equation Twenty-Seven

This furnishes what is possibly the shortest and most efficient route to ultra-accuracy CEIFs, for which there are of course no exact values.

By substitution of Equation Twenty-Five in Equation Twenty-Seven followed by simplifications we are able to write:-

$$K(k)=\frac{\pi}{2.AGM}\approx\frac{\pi}{2\left[2^{-\frac{5}{4}}.\left(1+\sqrt{\sin(\alpha)}\right).\left(1+\sin(\alpha)\right)^{\frac{1}{4}}.\left(\sqrt{\sin(\alpha)}\right)^{\frac{1}{4}}\right]}$$

$$\approx\frac{\pi}{\left[2^{-\frac{1}{4}}.\left(1+\sqrt{\sin(\alpha)}\right).\left(1+\sin(\alpha)\right)^{\frac{1}{4}}.\left(\sqrt{\sin(\alpha)}\right)^{\frac{1}{4}}\right]}$$

Equation Twenty-Eight

Equation Twenty-Eight rearranges to form:-

$$K(k)\approx\frac{\sqrt[4]{2}.\pi}{\left(1+\sqrt{\sin(\alpha)}\right).\left(1+\sin(\alpha)\right)^{\frac{1}{4}}.\left(\sqrt{\sin(\alpha)}\right)^{\frac{1}{4}}}\approx\frac{\sqrt[4]{2}.\pi}{\left(1+k^{\frac{1}{4}}\right).\left(1+\sqrt{k}\right)^{\frac{1}{4}}.k^{\frac{1}{16}}}$$

Equation Twenty-Nine

For K(½) the specific defect relative to the full AGM value of this CEIF is 0.000000000048646, that is to say ten-figure accuracy, as per the AGM approximation.

<u>Some Computational Structures</u>

Using an EXCEL® spreadsheet I compared the Hastings Polynomial and three of my own identities against a five-pass AGM CEIF for 101 values k = {0,0.01,1}.

For all 101 values of the five-pass AGM the residual error c_5 was smaller than $\pm 10^{-16}$, except for k = 0 for which it was 0.015625.

My own three identities elaborated were:-

$$K_{warren1}(k)=\frac{2^{\frac{1}{4}}.\pi}{\left(1+k^{\frac{1}{4}}\right).\left(1+\sqrt{k}\right)^{\frac{1}{4}}.k^{\frac{1}{16}}}$$

Equation Thirty a

$$K_{warren2}(k)=\frac{2^{\frac{1}{4}}.\pi}{\left(1+\sqrt{k^{\frac{1}{2}}}\right).\left(1+k^{\frac{1}{2}}\right)^{\frac{1}{4}}.\left(k^{\frac{1}{2}}\right)^{\frac{1}{8}}}$$

Equation Thirty b

$$K_{warren3}(k)=\frac{\sqrt{\sqrt{2}}.\pi}{\left(1+\sqrt{\sqrt{k}}\right).\sqrt{\sqrt{1+\sqrt{k}}}.\sqrt{\sqrt{\sqrt{\sqrt{k}}}}}$$

Equation Thirty c

For the AGM, Hastings and Warren1 relations a tabular summary is provided in Appendix One.

Figure One is a comparative plot of the Radix-16 logarithms of the absolute specific defects (relative to AGM) of the Hastings and Warren1 approximators.

It is manifest that though the Hastings Polynomial is cyclically unstable it is a superior estimator of CEIF from k = 0 to k = 0.5, when the Warren approximator becomes preferable.

From the point of view of verity the Warren2 and Warren3 structures are not significantly different to Warren1, though all three become numerically unstable at ~k>0.86 although not necessarily with defects coincident at the same arguments.

Such relations are illustrated by Figure Two.

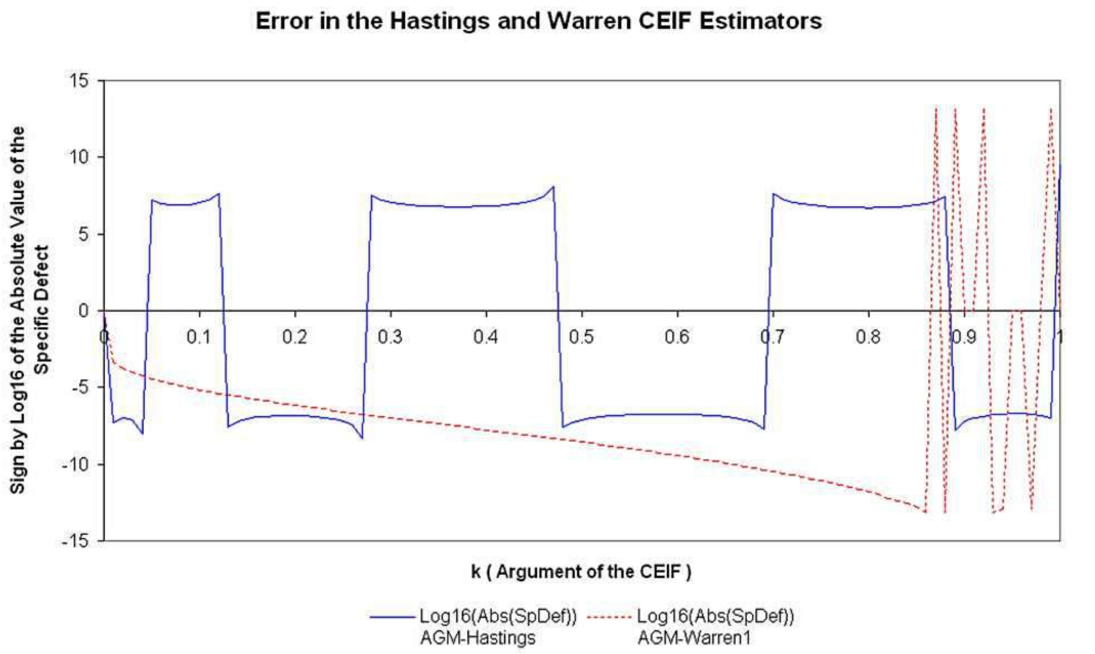

Figure One
Hastings and Warren1 Imprecisions
Relative to Fiducial Five-Pass AGMs

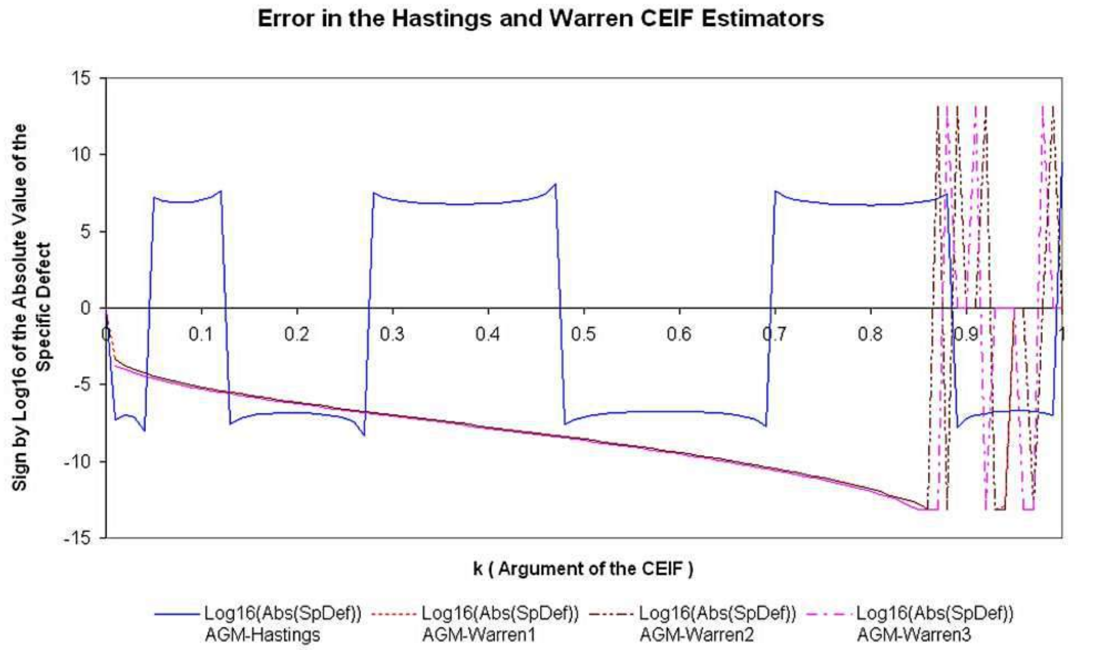

Figure Two
The Imprecisions of
Hastings and All Three Warren Estimators
Relative to Fiducial Five-Pass AGMs

It is clear from Equation Thirty c that the accuracy of the Warren estimator is driven by the precision and efficiency of whatever engine can be applied to the determination of repeated square roots.

Root Mean Square Errors

If u_i is some Estimate of a Particular CEIF and v_i is the relevant fiducial (i.e. full AGM) value at that point, then the Root Mean Square Error, RMS_{uv}, is given by:-

$$RMS_{uv} = \sqrt{\frac{\sum_{j=m}^{n} \left(\frac{u_i - v_i}{v_i} \right)^2}{(n-m+1)}}$$

Equation Thirty-One

The smaller the absolute value of RMS_{uv} the better.

Often it is simpler to compare the absolute logarithm of RMS when dealing with very tiny discrepancies. For example:-

$$RMS_{\log 10} = \left| \log_{10} \sqrt{\frac{\sum_{j=m}^{n} \left(\frac{u_i - v_i}{v_i} \right)^2}{(n - m + 1)}} \right|$$

Equation Thirty-Two

The bigger $RMS_{\log 10}$ the better.

Table Two shows the overall superiority of Hastings to Warren1 with the latter having an RMS error more than three orders of magnitude greater than the former.

Lower Bound:	0
Upper Bound:	1
Number of Intervals:	100
Interval:	0.01
RMS AGM-Hastings:	5.01914E-09
RMS AGM-Warren$_1$:	9.53785E-06

Table Two
Hastings and Warren1 RMS Errors

There is, however, a distinct reversal at k = 0.5 suggesting that the Warren estimator may be superior for K'(k), and this high-k benefit is shown in relief in Table Three.

		RMS	
		Hastings	**Warren1**
Series	**k**		
0-50	**0-0.5**	8.3865	4.8743
50-100	**0.5-1**	8.2428	10.9537

Table Three
Hastings and Warren1
Absolute Logarithm of RMS Errors

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AGM relations: Section 3.6 Page 9

APPENDIX ONE
A Tabular Summary of Certain Estimates of Complete Elliptic Integrals of the First Kind

Serial	k	$k^{0.5}$	α	AGM CEIF	Hastings CEIF	Warren $CEIF_1$	Warren $CEIF_2$	Fractional Specific Defect AGM-Hastings	Specific Defect Squared AGM-Hastings	Fractional Specific Defect AGM-Warren	Specific Defect Squared AGM-Warren1
0	0	0	0								
1	0.01	0.1	0.100167421	3.695637363	3.695637369	3.695967573	3.695967573	1.75035E-09	3.06374E-18	8.93514E-05	7.98366E-09
2	0.02	0.141421356	0.141897055	3.354141446	3.354141459	3.354239199	3.354239199	3.99416E-09	1.59533E-17	2.91441E-05	8.4938E-10
3	0.03	0.173205081	0.174083011	3.155874948	3.155874956	3.155917305	3.155917305	2.6985E-09	7.28188E-18	1.34218E-05	1.80143E-10
4	0.04	0.2	0.201357921	3.016112492	3.016112493	3.016134309	3.016134309	2.31076E-10	5.33963E-20	7.23345E-06	5.23228E-11
5	0.05	0.223606798	0.225513406	2.908337248	2.908337242	2.908349684	2.908349684	-2.14416E-09	4.5974E-18	4.27577E-06	1.82822E-11
6	0.06	0.244948974	0.247467063	2.820752497	2.820752486	2.820760079	2.820760079	-3.87396E-09	1.50076E-17	2.68793E-06	7.22498E-12
7	0.07	0.264575131	0.267763327	2.747073004	2.747072991	2.747077856	2.747077856	-4.79216E-09	2.29648E-17	1.76618E-06	3.11938E-12
8	0.08	0.282842712	0.286756552	2.683551406	2.683551393	2.683554627	2.683554627	-4.93277E-09	2.43322E-17	1.19997E-06	1.43993E-12
9	0.09	0.3	0.304692654	2.627773332	2.62777332	2.627775531	2.627775531	-4.42495E-09	1.95802E-17	8.36934E-07	7.00458E-13
10	0.1	0.316227766	0.321750554	2.578092113	2.578092104	2.57809365	2.57809365	-3.43421E-09	1.17938E-17	5.96181E-07	3.55432E-13
11	0.11	0.331662479	0.338065255	2.533334546	2.533334541	2.533335641	2.533335641	-2.1298E-09	4.53604E-18	4.3212E-07	1.86728E-13
12	0.12	0.346410162	0.353741606	2.492635323	2.492635322	2.492636115	2.492636115	-6.67288E-10	4.45273E-19	3.17786E-07	1.00988E-13
13	0.13	0.360555128	0.368862984	2.455338028	2.45533803	2.455338609	2.455338609	8.19888E-10	6.72216E-19	2.36596E-07	5.59775E-14
14	0.14	0.374165739	0.383497004	2.42093296	2.420932966	2.420933391	2.420933391	2.22369E-09	4.94479E-18	1.78013E-07	3.16887E-14
15	0.15	0.387298335	0.397699415	2.389016486	2.389016495	2.389016809	2.389016809	3.46165E-09	1.19831E-17	1.3516E-07	1.82683E-14
16	0.16	0.4	0.411516846	2.359263555	2.359263565	2.359263799	2.359263799	4.47537E-09	2.00289E-17	1.03439E-07	1.06997E-14
17	0.17	0.412310563	0.424988783	2.331408568	2.33140858	2.331408754	2.331408754	5.22798E-09	2.73318E-17	7.97132E-08	6.3542E-15
18	0.18	0.424264069	0.438149031	2.305231737	2.30523175	2.305231879	2.305231879	5.70138E-09	3.25057E-17	6.18046E-08	3.81981E-15
19	0.19	0.435889894	0.451026812	2.280549138	2.280549152	2.280549248	2.280549248	5.89316E-09	3.47293E-17	4.81776E-08	2.32108E-15
20	0.2	0.447213595	0.463647609	2.257205327	2.25720534	2.257205412	2.257205412	5.81373E-09	3.37995E-17	3.77341E-08	1.42386E-15
21	0.21	0.458257569	0.476033818	2.235067755	2.235067768	2.235067822	2.235067822	5.48362E-09	3.00701E-17	2.96791E-08	8.8085E-16
22	0.22	0.469041576	0.488205263	2.214022498	2.214022509	2.21402255	2.21402255	4.93095E-09	2.43142E-17	2.34309E-08	5.49009E-16
23	0.23	0.479583152	0.500179609	2.193970925	2.193970935	2.193970966	2.193970966	4.1892E-09	1.75494E-17	1.85595E-08	3.44454E-16
24	0.24	0.489897949	0.511972688	2.17482709	2.174827097	2.174827122	2.174827122	3.29532E-09	1.08591E-17	1.47439E-08	2.17384E-16
25	0.25	0.5	0.523598776	2.156515647	2.156515652	2.156515673	2.156515673	2.28798E-09	5.23486E-18	1.17432E-08	1.37902E-16
26	0.26	0.509901951	0.535070807	2.138970184	2.138970186	2.138970204	2.138970204	1.20618E-09	1.45487E-18	9.37446E-09	8.78805E-17
27	0.27	0.519615242	0.546400564	2.122131863	2.122131863	2.122131879	2.122131879	8.80284E-11	7.749E-21	7.49847E-09	5.62271E-17
28	0.28	0.529150262	0.557598827	2.10594832	2.105948318	2.105948333	2.105948333	-1.03022E-09	1.06135E-18	6.00831E-09	3.60998E-17
29	0.29	0.538516481	0.568675503	2.090372747	2.090372742	2.090372757	2.090372757	-2.11494E-09	4.47296E-18	4.82147E-09	2.32466E-17
30	0.3	0.547722558	0.57963974	2.075363135	2.075363129	2.075363143	2.075363143	-3.13576E-09	9.83299E-18	3.87398E-09	1.50077E-17
31	0.31	0.556776436	0.590500015	2.060881647	2.060881638	2.060881653	2.060881653	-4.066E-09	1.65324E-17	3.11598E-09	9.70931E-18
32	0.32	0.565685425	0.601264217	2.046894077	2.046894067	2.046894082	2.046894082	-4.88297E-09	2.38434E-17	2.50845E-09	6.29232E-18
33	0.33	0.574456265	0.611939715	2.033369409	2.033369398	2.033369413	2.033369413	-5.56814E-09	3.10042E-17	2.02073E-09	4.08336E-18
34	0.34	0.583095189	0.62253342	2.020279429	2.020279416	2.020279432	2.020279432	-6.10727E-09	3.72988E-17	1.62865E-09	2.6525E-18
35	0.35	0.591607978	0.633051836	2.007598398	2.007598385	2.007598401	2.007598401	-6.49033E-09	4.21243E-17	1.31307E-09	1.72415E-18
36	0.36	0.6	0.643501109	1.995302778	1.995302764	1.99530278	1.99530278	-6.71142E-09	4.50432E-17	1.05881E-09	1.12108E-18
37	0.37	0.608276253	0.653887062	1.98337098	1.983370966	1.983370981	1.983370981	-6.76864E-09	4.58145E-17	8.53789E-10	7.28955E-19
38	0.38	0.6164414	0.664215238	1.971783162	1.971783149	1.971783163	1.971783163	-6.66382E-09	4.44065E-17	6.88362E-10	4.73842E-19
39	0.39	0.6244998	0.674490928	1.960521044	1.960521032	1.960521045	1.960521045	-6.40229E-09	4.09893E-17	5.54821E-10	3.07826E-19
40	0.4	0.632455532	0.684719203	1.94956775	1.949567738	1.949567751	1.949567751	-5.99256E-09	3.59108E-17	4.46986E-10	1.99796E-19
41	0.41	0.640312424	0.694904938	1.938907665	1.938907655	1.938907666	1.938907666	-5.44598E-09	2.96586E-17	3.59895E-10	1.29524E-19
42	0.42	0.64807407	0.705052837	1.928526318	1.928526309	1.928526319	1.928526319	-4.77638E-09	2.28138E-17	2.89559E-10	8.38442E-20
43	0.43	0.655743852	0.715167456	1.918410269	1.918410261	1.91841027	1.91841027	-3.99974E-09	1.59979E-17	2.32763E-10	5.41786E-20
44	0.44	0.663324958	0.725253222	1.908547016	1.90854701	1.908547017	1.908547017	-3.13376E-09	9.82044E-18	1.86915E-10	3.49372E-20
45	0.45	0.670820393	0.735314453	1.89892491	1.898924906	1.898924911	1.898924911	-2.19748E-09	4.82892E-18	1.49922E-10	2.24765E-20
46	0.46	0.678232998	0.745355373	1.889533079	1.889533077	1.889533079	1.889533079	-1.21093E-09	1.46635E-18	1.20091E-10	1.44219E-20
47	0.47	0.68556546	0.755380134	1.88036136	1.880361359	1.88036136	1.88036136	-1.94719E-10	3.79154E-20	9.60547E-11	9.2265E-21
48	0.48	0.692820323	0.765392826	1.87140024	1.871400241	1.87140024	1.87140024	8.30324E-10	6.89438E-19	7.67049E-11	5.88365E-21
49	0.49	0.7	0.775397497	1.862640802	1.862640806	1.862640802	1.862640802	1.84351E-09	3.39853E-18	6.11444E-11	3.73863E-21
50	0.5	0.707106781	0.785398163	1.854074677	1.854074683	1.854074677	1.854074677	2.82463E-09	7.97854E-18	4.86462E-11	2.36645E-21

Table A1
CEIF Estimates
Between k = 0 and k = 0.5

Serial	k	$k^{0.5}$	α	AGM CEIF	Hastings CEIF	Warren $CEIF_1$	Warren $CEIF_2$	Fractional Specific Defect AGM-Hastings	Specific Defect Squared AGM-Hastings	Fractional Specific Defect AGM-Warren	Specific Defect Squared AGM-Warren1
50	0.5	0.707106781	0.785398163	1.854074677	1.854074683	1.854074677	1.854074677	2.82463E-09	7.97854E-18	4.86462E-11	2.36645E-21
51	0.51	0.714142843	0.79539883	1.845693998	1.845694005	1.845693998	1.845693998	3.75429E-09	1.40947E-17	3.86214E-11	1.49161E-21
52	0.52	0.721110255	0.805403501	1.837491363	1.837491372	1.837491363	1.837491363	4.6142E-09	2.12909E-17	3.05931E-11	9.35939E-22
53	0.53	0.728010989	0.815416193	1.829459799	1.829459808	1.829459799	1.829459799	5.38747E-09	2.90249E-17	2.41744E-11	5.84404E-22
54	0.54	0.734846923	0.825440953	1.821592727	1.821592738	1.821592727	1.821592727	6.05886E-09	3.67098E-17	1.90526E-11	3.63E-22
55	0.55	0.741619849	0.835481874	1.813883937	1.813883949	1.813883937	1.813883937	6.61502E-09	4.37585E-17	1.49732E-11	2.24196E-22
56	0.56	0.748331477	0.845543105	1.806327559	1.806327572	1.806327559	1.806327559	7.04467E-09	4.96274E-17	1.17318E-11	1.37635E-22
57	0.57	0.754983444	0.855628871	1.798918039	1.798918052	1.798918039	1.798918039	7.33882E-09	5.38582E-17	9.16287E-12	8.39583E-23
58	0.58	0.761577311	0.86574349	1.791650117	1.79165013	1.791650117	1.791650117	7.49086E-09	5.6113E-17	7.13135E-12	5.08562E-23
59	0.59	0.768114575	0.875891389	1.784518805	1.784518818	1.784518805	1.784518805	7.49672E-09	5.62008E-17	5.53021E-12	3.05833E-23
60	0.6	0.774596669	0.886077124	1.777519371	1.777519385	1.777519371	1.777519371	7.35494E-09	5.40951E-17	4.27145E-12	1.82453E-23
61	0.61	0.781024968	0.896305399	1.770647323	1.770647336	1.770647323	1.770647323	7.06669E-09	4.99381E-17	3.28531E-12	1.07933E-23
62	0.62	0.787400787	0.906581089	1.763898389	1.763898401	1.763898389	1.763898389	6.63584E-09	4.40343E-17	2.51564E-12	6.32846E-24
63	0.63	0.793725393	0.916909265	1.757268505	1.757268516	1.757268505	1.757268505	6.0689E-09	3.68315E-17	1.9171E-12	3.67528E-24
64	0.64	0.8	0.927295218	1.750753803	1.750753812	1.750753803	1.750753803	5.37499E-09	2.88905E-17	1.45358E-12	2.11288E-24
65	0.65	0.806225775	0.93774449	1.744350597	1.744350605	1.744350597	1.744350597	4.56576E-09	2.08462E-17	1.096E-12	1.20121E-24
66	0.66	0.81240384	0.948262907	1.738055373	1.73805538	1.738055373	1.738055373	3.65526E-09	1.33609E-17	8.21718E-13	6.7522E-25
67	0.67	0.818535277	0.958856612	1.731864778	1.731864783	1.731864778	1.731864778	2.65977E-09	7.07438E-18	6.12209E-13	3.748E-25
68	0.68	0.824621125	0.96953211	1.72577561	1.725775612	1.72577561	1.72577561	1.59766E-09	2.55253E-18	4.53025E-13	2.05231E-25
69	0.69	0.830662386	0.980296312	1.719784808	1.719784809	1.719784808	1.719784808	4.89136E-10	2.39254E-19	3.32721E-13	1.10703E-25
70	0.7	0.836660027	0.991156586	1.713889448	1.713889447	1.713889448	1.713889448	-6.44033E-10	4.14778E-19	2.42529E-13	5.88202E-26
71	0.71	0.842614977	1.002120823	1.708086731	1.708086728	1.708086731	1.708086731	-1.77877E-09	3.16402E-18	1.75365E-13	3.07528E-26
72	0.72	0.848528137	1.0131975	1.702373977	1.702373972	1.702373977	1.702373977	-2.89104E-09	8.35809E-18	1.25867E-13	1.58426E-26
73	0.73	0.854400375	1.024395763	1.69674862	1.696748613	1.69674862	1.69674862	-3.95619E-09	1.56515E-17	8.93806E-14	7.9889E-27
74	0.74	0.860232527	1.03572552	1.691208199	1.691208191	1.691208199	1.691208199	-4.9494E-09	2.44966E-17	6.27583E-14	3.9386E-27
75	0.75	0.866025404	1.047197551	1.685750355	1.685750345	1.685750355	1.685750355	-5.84602E-09	3.4176E-17	4.37306E-14	1.91236E-27
76	0.76	0.871779789	1.058823639	1.680372823	1.680372812	1.680372823	1.680372823	-6.62213E-09	4.38526E-17	2.99958E-14	8.99748E-28
77	0.77	0.877496439	1.070616718	1.675073429	1.675073417	1.675073429	1.675073429	-7.25496E-09	5.26344E-17	2.02814E-14	4.11335E-28
78	0.78	0.883176087	1.082591063	1.669850086	1.669850073	1.669850086	1.669850086	-7.72347E-09	5.96519E-17	1.35632E-14	1.83961E-28
79	0.79	0.888819442	1.094762509	1.664700786	1.664700773	1.664700786	1.664700786	-8.00889E-09	6.41423E-17	8.93673E-15	7.98652E-29
80	0.8	0.894427191	1.107148718	1.659623599	1.659623585	1.659623599	1.659623599	-8.09533E-09	6.55344E-17	5.75306E-15	3.30977E-29
81	0.81	0.9	1.119769515	1.654616668	1.654616654	1.654616668	1.654616668	-7.9704E-09	6.35274E-17	3.75752E-15	1.41189E-29
82	0.82	0.905538514	1.132647296	1.649678205	1.649678193	1.649678205	1.649678205	-7.62589E-09	5.81541E-17	1.88438E-15	3.5509E-30
83	0.83	0.911043358	1.145807544	1.644806491	1.644806479	1.644806491	1.644806491	-7.05841E-09	4.98211E-17	1.21498E-15	1.47617E-30
84	0.84	0.916515139	1.159279481	1.639999866	1.639999856	1.639999866	1.639999866	-6.27019E-09	3.93153E-17	8.12358E-16	6.59926E-31
85	0.85	0.921954446	1.173096912	1.635256732	1.635256724	1.635256732	1.635256732	-5.26979E-09	2.77707E-17	4.07357E-16	1.6594E-31
86	0.86	0.92736185	1.187299323	1.630575549	1.630575542	1.630575549	1.630575549	-4.07291E-09	1.65886E-17	1.36176E-16	1.85438E-32
87	0.87	0.932737905	1.201933343	1.625954829	1.625954825	1.625954829	1.625954829	-2.70317E-09	7.30714E-18	-1.36563E-16	1.86493E-32
88	0.88	0.938083152	1.217054721	1.621393138	1.621393136	1.621393138	1.621393138	-1.19301E-09	1.42328E-18	1.36947E-16	1.87544E-32
89	0.89	0.943398113	1.232731072	1.616889091	1.616889091	1.616889091	1.616889091	4.15452E-10	1.726E-19	-1.37328E-16	1.88591E-32
90	0.9	0.948683298	1.249045772	1.612441349	1.612441352	1.612441349	1.612441349	2.06954E-09	4.28298E-18	0	0
91	0.91	0.953939201	1.266103673	1.60804862	1.608048626	1.60804862	1.60804862	3.70502E-09	1.37272E-17	0	0
92	0.92	0.959166305	1.284039775	1.603709655	1.603709663	1.603709655	1.603709655	5.24521E-09	2.75123E-17	-1.38457E-16	1.91703E-32
93	0.93	0.964365076	1.303033	1.599423245	1.599423255	1.599423245	1.599423245	6.59992E-09	4.35589E-17	1.38828E-16	1.92732E-32
94	0.94	0.969535971	1.323329264	1.595188221	1.595188234	1.595188221	1.595188221	7.66445E-09	5.87438E-17	2.78393E-16	7.75027E-32
95	0.95	0.974679434	1.345282921	1.591003454	1.591003467	1.591003454	1.591003454	8.31856E-09	6.91984E-17	0	0
96	0.96	0.979795897	1.369438406	1.586867847	1.586867861	1.586867847	1.586867847	8.42536E-09	7.09867E-17	0	0
97	0.97	0.98488578	1.396713316	1.582780342	1.582780355	1.582780342	1.582780342	7.83023E-09	6.13125E-17	2.80575E-16	7.87225E-32
98	0.98	0.989949494	1.428899272	1.578739912	1.578739922	1.578739912	1.578739912	6.35966E-09	4.04453E-17	0	0
99	0.99	0.994987437	1.470628906	1.574745562	1.574745568	1.574745562	1.574745562	3.82013E-09	1.45934E-17	-1.41003E-16	1.9882E-32
100	1	1	1.570796327	1.570796327	1.570796327	1.570796327	1.570796327	-3.11723E-12	9.7171E-24	0	0

Table A2
CEIF Estimates
Between k = 0.5 and k = 1