The Estimation of Vertex Concentrations in Delaunay Triangulations Whether Limited or Sufficient Data is Available

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The value of the Areal Density of plotted points is a statistic of prime interest in the study of vertices upon the Euclidean Plane, E^2.

Miles[1] calls the Concentration ρ in his 1970 discussion of the statistical characteristics of the Delaunay Triangulations of randomly-scattered points. Under such arbitrary conditions he and others identify ρ with the Poissonian Intensity Parameter, λ, essentially a mean areal concentration varying from ρ only by a 2π scaling factor.

When, however, we require to estimate ρ for a given triangulation we are confronted with the finitude of a particular convex hull whose boundary-vertices make a merely partial contribution to the vertex sum. It is the *effective* vertex sum, which we must divide by hull area in order to estimate concentration.

It is possible, though troublesome and potentially expensive, to implement an algorithm which identifies boundary vertices and counts them. This is sufficient to compute a contribution to concentration in terms of included angles at boundary vertices.

I shall show that even when a hull is both "rotund" and highly-populated with vertices, the effect of boundary-vertex "shares" upon calculations of areal point density is never negligible.

PART I

ESTIMATION WHEN THE NUMBER OF BOUNDARY POINTS IS UNKNOWN

A useful property of Delaunay tessellations is that their component triangles are "locally equilateral". What this means is that the Delaunay geometry optimises triangulations in such a way that every internal angle of each triangular cell is "as near as practicable" to π/3 radians. Indeed, Miles has proposed a probability density function for these internal angles which generates an expected mean angle of π/3 radians.

A Delaunay Triangulation shares this "locally equilateral" property with its structural epitome, the regular hexagon, because any regular hexagon can be constructed from a finite number of (perfectly) equilateral triangles, as is that of Figure One.

I accordingly propose an acceptable adjustor for Delaunay areal concentration estimates based upon the point geometry of the regular hexagon.

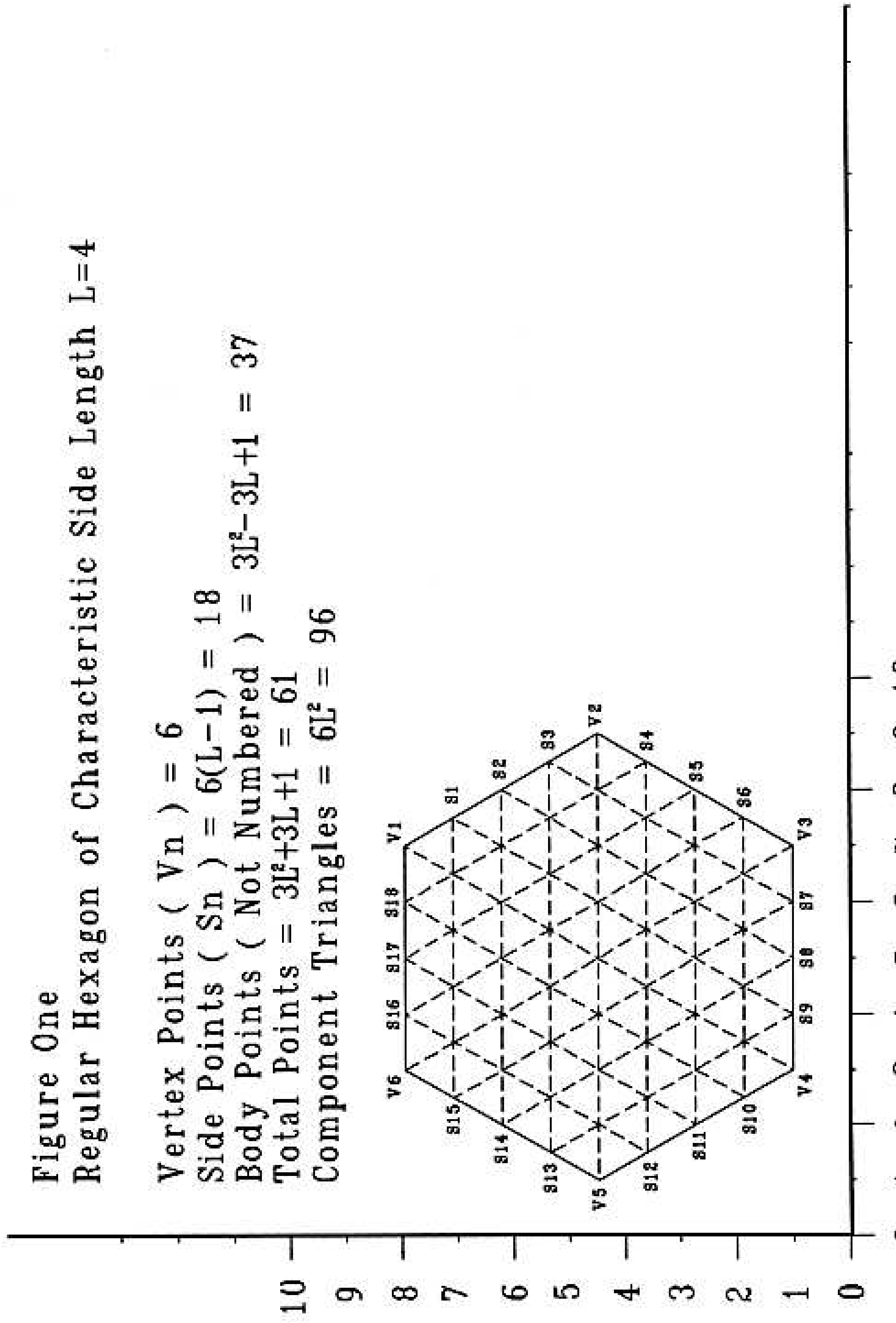
Figure One
Regular Hexagon of Characteristic Side Length L=4
Vertex Points (Vn) = 6
Side Points (Sn) = 6(L−1) = 18
Body Points (Not Numbered) = 3L²−3L+1 = 37
Total Points = 3L²+3L+1 = 61
Component Triangles = 6L² = 96
V1
V2
V3
V4
V5
V6
S1
S2
S3
S4
S5
S6
S7
S8
S9
S10
S11
S12
S13
S14
S15
S16
S17
S18
0 1 2 3 4 5 6 7 8 9 10
0 1 2 3 4 5 6 7 8 9 10

Body and Boundary Points

Any non-degenerate triangulation will present points wholly included within the convex hull which we will call BODY POINTS, and points along the hull perimeter which are BOUNDARY POINTS. Each body point contributes 360°-worth of mass potential to the areal density, but as aforementioned a boundary point shares its mass potential between the hull interior and the spatial environment. That potential assignable to the space outwith is not properly employed in measuring the areal density of the point cloud inside the hull.

It is convenient to assess a point's MASS POTENTIAL in terms of the angular fraction of its orbit within the hull. For example, a regular hexagon has VERTEX POINTS which each have a mass potential of _ (because $2\pi/3$ radians are included) and SIDE POINTS which have a potential of ½. In addition to these BOUNDARY POINTS there are usually BODY POINTS which each have a potential of 1.

All relevant aspects of hexagon geometry may be referred to The Characteristic Side Length, L. Side Length is integral for regular hexagons but the concept of L can be extended to characterise arbitrary hulls, when L may be a fraction. The actual physical size of L is irrelevant to this discussion.

Points in a Regular Hexagon

Number of Hexagon Vertex Points

The Number of Hexagon Vertex Points, v, is:-

$$v = 6 \qquad \textbf{Eqn.1}$$

Number of Hexagon Side Points

Simple analysis reveals that:-

$$s = 6(L - 1) \qquad \textbf{Eqn.2}$$

Number of Hexagon Body Points

Inspection of the internal structure of hexagons compounded of equilateral triangles demonstrates that the Number of Body Points, f, is given by:-

$$\begin{aligned} f &= 2\sum_{i=L}^{L+L-2} i + 2L - 1 \\ &= 2\sum_{i=1}^{2L-2} i - 2\sum_{i=1}^{L-1} i + 2L - 1 \\ &= 2.\frac{1}{2}(2L-2)[(2L-2)+1] - 2.\frac{1}{2}(L-1)[(L-1)+1] + 2L - 1 \qquad \textbf{Eqn.3} \\ &= (2L-2)(2L-1) - L(L-1) + 2L - 1 \\ &= 4L^2 - 6L + 2 - L^2 + L + 2L - 1 \\ &= 3L^2 - 3L + 1 \end{aligned}$$

Number of Hexagon Boundary Points

Addition of Equations One and Two shows that a regular hexagon has a Number of Boundary Points, b, given by:-

$$b = 6L \qquad \textbf{Eqn.4}$$

Total Points in a Regular Hexagon

The Total Points in a Regular Hexagon is given by:-

$$V = b + f$$
$$\therefore V = 6L + 3L^2 - 3L + 1$$

Therefore:-

$$V = 3L^2 + 3L + 1 \qquad \textbf{Eqn.5}$$

The Number of Component Equilateral Triangles, T

The Number of Component Equilateral Triangles, T, is given by:-

$$T = 2\left[L + 2L + 2\sum_{i=L+1}^{2L-1} i \right]$$
$$= 2\left[3L + 2\sum_{i=1}^{2L-1} i - 2\sum_{i=1}^{L} i \right]$$
$$= 2\left[3L + 2.\frac{1}{2}(2L-1)[(2L-1)+1] - 2.\frac{1}{2}L(L+1) \right]$$
$$= 2[3L + 2L(2L-1) - L(L+1)]$$
$$= 2[3L + 4L^2 - 2L - L^2 - L]$$
$$= 2[3L^2]$$

Therefore:-

$$T = 6L^2 \qquad \textbf{Eqn.6}$$

<u>The Boundary is Always Significant</u>

The mass potential contribution of boundary points is always considerable for a practical triangulation on E^2.

In order to demonstrate this, propose that should only 1% of points of a regular hexagon be boundary points, then we will use a simple count of tessellation vertices to estimate ρ. (i.e. The discrepancy of estimation is negligible with 99% of points within the hull).

This postulate may be written:-

$$0.01 = \frac{b}{V} = \frac{6L}{3L^2 + 3L + 1} \qquad \textbf{Eqn.7}$$

which may be rearranged as:-

$$0 = 0.03L^2 - 5.97L + 0.01 \qquad \textbf{Eqn.8}$$

The quadratic roots of Equation Eight are:-

L = 0.00167505597 and L = 198.998324944

From which it follows (technically speaking) that the boundary is negligible for degenerate hexagons with between 1 and 1.00503358537 points: And for regular hexagons with more than 119398.994966 (i.e. 119399) points.

For everything in between, the triangulations that get performed, more than 1% of points lie on the boundary.

Determination of the Characteristic Length

Most triangulations are not of course regular hexagons and may have any whole number of triangulation points.

Equation Five may be re-arranged as a quadratic and solved via the Newton-Raphson Method to determine a general Characteristic Length which may well be a fraction:-

$$V = 3L^2 + 3L + 1$$

$$\therefore 1 = \frac{3}{V}.L^2 + \frac{3}{V}.L + \frac{1}{V}$$

$$\therefore 0 = \frac{3}{V}.L^2 + \frac{3}{V}.L + \frac{1}{V} - 1$$

Accordingly the Newton-Raphson Function, f(L), is:-

$$f(L) = \frac{3}{V}.L^2 + \frac{3}{V}.L + \frac{1}{V} - 1 \qquad \textbf{Eqn.9}$$

and the Differential of f(L), g(L) is:-

$$g(L) = \frac{6}{V}.L + \frac{3}{V} \qquad \textbf{Eqn.10}$$

Example for One Hundred Points

A point field contains a hundred dots for triangulation. What is its characteristic length?:-

$$f(L) = \frac{3}{V}.L^2 + \frac{3}{V}.L + \frac{1}{V} - 1$$

$$= 0.03L^2 + 0.03L - 0.99$$

and:-

$$g(L) = \frac{6}{V}.L + \frac{3}{V} = 0.06L + 0.03$$

By tabulation we can determine that L is between 5 and 6. Our initial estimate of L, x_0, is set to 5.5.

After three iterations x_3=L converges to 5.26628129734.

Back substitution into Equation Five computes V=100.

An Application to the Delaunay Test Field, SLOAN1.VAL

Figure Two illustrates a six-point test field. The areal density of the points within the hull is to be estimated. This will of course require the hull area as a divisor, but our interest is presently confined to the point mass potential which bears within the hull V4-V6-V5-V2-V1.

We may discount V=6 straight away since all five hull points contribute only a minority of their potentials to the interior.

An exact computation of point potential is achieved by summing all the internal angles of the pentagon and dividing by 360°:-

Vertex	Angle°
V4	135
V6	97.125
V5	112.62
V2	90
V1	105.255
	540°

Which is 540°/360°=1.5 units. We must remember that there is V3 in the midst of the hull, which alone is worth 1 unit giving a grand total of V_e=2.5 units of effective points.

Would regular hexagon theory estimate this V with sufficient accuracy and without the cost and complication of counting algorithms?

First, note that there are V=6 physical points in the test field. Apply Newton-Raphson through four iterations with x_0=0.5 to obtain L=0.88443731048.

Now tabulate the fractional vertices of each species to estimate the total V_e' for this data:-

SPECIES	EQUATION	COMPUTED POINTS	WEIGHT	SUBTOTAL
Vertex	6	6	_	2.
Side	6(L-1)	-0.69337613708	½	-0.34668806854
Body	$3L^2$-3L+1	+0.69337613708	1	+0.69337613708
Total	$3L^2$+3L+1		V_e' =	2.34668806854

V_e' is notably close to the measured value of total effective potential. The specific defect is:-

$$\varepsilon\% = 100\left(\frac{V_e - V_{e'}}{V_{e'}}\right) = 6.53311931459\% \qquad \textbf{Eqn.11}$$

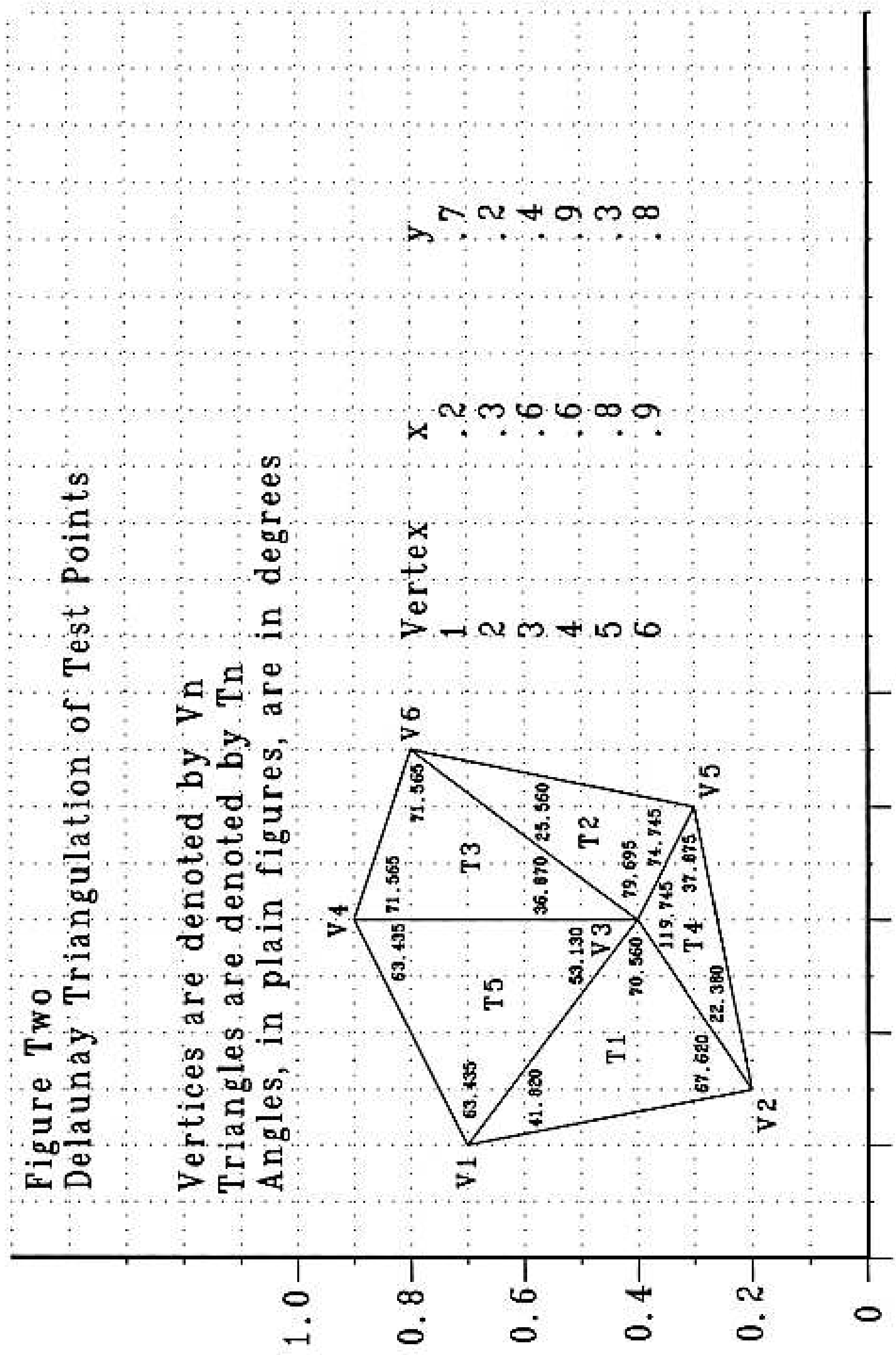
Figure Two
Delaunay Triangulation of Test Points
Vertices are denoted by Vn
Triangles are denoted by Tn
Angles, in plain figures, are in degrees
Vertex x y
1 .2 .7
2 .3 .2
3 .6 .4
4 .6 .9
5 .8 .3
6 .9 .8
V1
V2
V3
V4
V5
V6
T1
T2
T3
T4
T5
63.435
41.820
67.620
22.380
70.560
53.130
63.435
71.565
36.870
71.565
25.560
79.695
74.745
37.875
119.745
0
0.2
0.4
0.6
0.8
1.0

PART II

ESTIMATION WHEN THE NUMBER OF BOUNDARY POINTS IS KNOWN

Suppose that we can count the actual number of triangulation points which lie upon the convex hull envelope of a certain Delaunay Triangulation.

Let us call this count N_S. Then the hull constitutes an N_S-gon with N_S internal angles and N_S sides. Each ith vertex of this polygon has an Internal Angle, α_i, and:-

$$\sum \alpha_i = \pi(N_S - 2) \qquad \textbf{Eqn.12}$$

and the Mass Potential Contribution, V_e, due to these boundary points is:-

$$V_e = \frac{N_S - 2}{2} \qquad \textbf{Eqn.13}$$

Accordingly:-

$$\begin{aligned} N_e &= N_b + V_e \\ &= (N - N_S) + V_e \end{aligned}$$

or:-

$$N_e = (N - N_S) + \frac{N_S - 2}{2}$$

which reduces to:-

$$N_e = N - \frac{N_S}{2} - 1 \qquad \textbf{Eqn.14}$$

PART III

THE COMPUTATION OF HULL AREA

A_H is the Delaunay Triangulation Hull Area in square units. It may be convenient individually to compute the component triangle areas and sum them for hull area. But it is also of course possible to apply the formula for the area of a general polygon:-

$$A_H = \left| \frac{1}{2} \left\{ x_n . y_1 - y_n . x_i + \sum_{i=1}^{n-1} x_i \bullet y_{i+1} - \sum_{i=1}^{n-1} y_i \bullet x_{i+1} \right\} \right| \qquad \textbf{Eqn.15}$$

Allowing $n \equiv N_S$ for notational simplicity.

It then follows in general that:-

$$\rho' = \frac{N_e}{A_H} \quad \textbf{Eqn.16}$$

Notation

α_i	The Internal Angle at Hull Vertex i
A_H	The Delaunay Triangulation Hull Area
b	The Number of (Hexagon) Boundary Points
ε%	The Specific Defect
E^2	The Euclidean Plane (continuum)
f	The Number of Hexagon Body Points
f(L)	The Newton-Raphson Quadratic in L
g(L)	The Differential of f(L)
L	The Characteristic Length
n	The Number of Vertex Points
N	The Number of Triangulation Points
N_b	The Number of Body Points within a General Hull
N_e	The Effective Total Point Potential of a General Hull
N_S	The Number of Hull Boundary Points
π	The Ludolphine Constant
ρ	The (Ideal) Areal Concentration of Points
ρ'	The Computed Areal Concentration of Points
s	The Number of Hexagon Side Points
T	The Number of Component Equilateral Triangles
v	The Number of Hexagon Vertex Points
V	The Total Number of Points
V_e	The Measured Effective Total Point Potential
V_e'	The Estimated Effective Total Point Potential
x_0	The Newton-Raphson Starter Guess
x_i	The x Co-ordinate of a Hull Vertex
x_n	a Newton-Raphson Estimate
y_i	The y Co-ordinate of a Hull Vertex

Reference

1 "On the homogenous planar Poisson point-process"
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