An Enhanced Estimator for Factorials and for The Gamma Function

by
James R Warren BSc MSc PhD PGCE MIEEE MACM

On page 84 of Sprecher[1] the following error bounds for Stirling's Formula are presented:-

$$\sqrt{2\pi}.n^{n+\frac{1}{2}}.e^{-n} < n! < \sqrt{2\pi}.n^{n+\frac{1}{2}}.e^{-n}\left(1+\frac{1}{4n}\right) \qquad \textbf{Eqn.1}$$

where the lower bound is the form of Stirling's Formula for the estimation of higher factorials which is usually quoted in the literature.

Let E, the Central Expectation of the Factorial, be defined as:-

$$E = \sqrt{2\pi}.n^{n+\frac{1}{2}}.e^{-n}\left(1+\frac{1}{8n}\right) \qquad \textbf{Eqn.2}$$

Further define the Relative Error in Estimation, D, as:-

$$D = \frac{E - n!}{E} = 1 - \frac{n!}{E} \qquad \textbf{Eqn.3}$$

n, n! and D were generated for $1 \le n \le 100$ using an EXCEL spreadsheet. The following power law fitment was found to apply:-

$$D = 0.03886870\, n^{-0.98356437} \qquad \textbf{Eqn.4}$$

with an r^2 determination coefficient of 0.99987222.

Accordingly we may approximate this Relative Error in these terms:-

$$D \approx \frac{1}{8\pi n} \qquad \textbf{Eqn.5}$$

Now because:-

$$n! = E\ (1 - D) \qquad \textbf{Eqn.6}$$

any factorial may be approximated as:-

$$n! \approx \sqrt{2\pi}.n^{n+\frac{1}{2}}.e^{-n}\left(1+\frac{1}{8n}\right)\left(1-\frac{1}{8\pi n}\right) \qquad \textbf{Eqn.7}$$

The enhanced Stirling's Formula of Equation Seven has a maximum relative error of 0.388792% for 1! reducing to 0.02076% for 3! and stabilising near to -0.00179% at 100!. The Stirling's Formula relative error is 21.7 times larger for 1! and 46.6 times greater at 100!.

For the range $1 \le n \le 99$ the Root Mean Square (RMS) errors of the (monotonically-increasing) deviations from the true factorial are 7.8924×10^{151} for Stirling's Formula and 1.6973×10^{150} for the Enhanced Stirling's Formula of Equation Seven, yielding an $RMS_{Stirling}/RMS_{Eqn.7}$ ratio of 46.50 which improves to 78.97 for $1 \le n \le 10$.

Generalisation for The Gamma Function

It may naturally be wondered whether Equation Seven is capable of being generalised for the gamma function in view of the apparent absence of a computational impediment to n adopting fractional values.

Let such a General Value for n be an integral or fractional real number called f.

Numerical experiments have confirmed that:-

$$\Gamma(f+1) \approx \sqrt{2\pi}.f^{f+\frac{1}{2}}.e^{-f}\left(1+\frac{1}{8f}\right)\left(1-\frac{1}{8\pi f}\right) \qquad \textbf{Eqn.8}$$

Allow that A is the Estimate of Γ(f+1) yielded by an EXCEL Spreadsheet, which is assumed fiducial. Further, B is the approximation given by the RHS of Equation Eight.

An adequate and simple measure of Relative Error for this case is afforded by:-

$$\varepsilon = \frac{A}{B} \qquad \textbf{Eqn.9}$$

Now for fractional f=0.1,0.1,0.9 the Mean Relative Error ε_μ was 0.0426514 (say 4%) and for f=9,0.1,9.9 ε_μ was -0.000105141. Note, however, the sign change, which occurs when $4.15<f<4.16$.

Because the gamma function is subject to the recursive relation:-

$$\frac{\Gamma(f+1)}{\Gamma(f)} = f \qquad \textbf{Eqn.10}$$

it makes sense to compute Γ(f+1) on the interval $4 \le f \le 5$, and recursively to use Equation Ten to establish the function at the desired value.

For example, I wish to have Γ(0.3) which Equation Eight would infer to be 0.897470696. Now for Γ(4.3) the Equation Eight Estimate is 38.07849.

Hence:-

$$\Gamma(f) = \frac{\Gamma(f+1)}{f+1} \qquad \textbf{Eqn.11}$$

so:-

$$\Gamma(3.3) = \frac{\Gamma(4.3)}{4.3} = \frac{38.07849}{4.3} = 8.8554627907$$

and:-

$$\Gamma(2.3) = \frac{\Gamma(3.3)}{3.3} = 2.68347357294$$

and:-

$$\Gamma(1.3) = \frac{\Gamma(2.3)}{2.3} = 1.16672764041$$

and:-

$$\Gamma(0.3) = \frac{\Gamma(1.3)}{1.3} = 0.89748280031$$

Now using Equation Eight the ε for f=0.3 is 1.0305023688 or about 3% error. When, however, we use Equation Eight to estimate Γ(4.3) and then the recursion to define Γ(0.3) the ε reduces to 0.99998651304, an absolute error of about 0.00135%.

A Simplified Approximation for the Gamma Function

Combination of Equations Eight and Ten allows us to write:-

$$\Gamma(f) \approx \sqrt{2\pi} . f^{f-\frac{1}{2}} . e^{-f} \left(1 + \frac{1}{8f}\right)\left(1 - \frac{1}{8\pi f}\right) \qquad \textbf{Eqn.12}$$

which allows direct estimation of Gamma for any valid argument.

Now since:-

$$\pi = (\Gamma(0.5))^2 \qquad \textbf{Eqn.13}$$

it is easily computed that a fiducial value of Γ(0.5) is 1.77245385091.

Equation Twelve estimates Γ(0.5) as 1.74920192351 so that the relative error is about 1.3%, typical for $0<f\leq 1$.

EXCEL, working to its maximum 15-digit precision, provides an estimate for Γ(4.15) of 7.26687266072991. Equation Twelve estimates this value as 7.26687112188 giving an error of approximately 0.000021176% near to the point where the sign of the error changes.

Acknowledgement

I am indebted to Mr Rupert Osborne for the loan of literature upon real analysis and in particular for the error range of Equation One.

Reference

1 "Elements of Real Analysis"
David A Sprecher
Dover Publications of New York 1970
ISBN 0-486-65385-4 (pbk)

Notation

A	The Fiducial Estimate of Γ(f+1)
B	The Equation Eight Approximation of Γ(f+1)
Γ	The Gamma Function
Γ(*)	The Gamma Function of Argument *
D	The Relative Error
ε	The Fractional Error
ε_μ	The Mean Fractional Error
e	The Napierian Base
E	The Central Expectation of the Factorial
f	The Number whose Gamma Function is to be found
n	The Integer whose Factorial is to be Estimated
π	The Ludolphine Constant
r^2	The Coefficient of Determination