Some Further Enhancement of a Gamma Function Estimator

by

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At the end of my June 1998 paper "An Enhanced Estimator for Factorials and for The Gamma Function" I proposed the following estimator:-

$$W_0 = \Gamma(f) \approx \sqrt{2\pi}\,.\,f^{\,f-\frac{1}{2}}.e^{-f}\left(1+\frac{1}{8f}\right)\left(1-\frac{1}{8\pi f}\right) \qquad \textbf{Eqn.1}$$

and described its especial precision in the region of f=4.3.

I subsequently essayed several avenues of improvement. The fitment of successive power-laws was particularly disappointing as errors re-advanced after the third or fourth power-law fitment to quotient residuals.

Nevertheless, I can propose two cubic equations of form $^3P(f^{-1})$ which I fitted to Equation One such that:-

$$W_1 \approx \Gamma(f) \approx {}^3P(f^{-1}).\sqrt{2\pi}\,.\,f^{\,f-\frac{1}{2}}.e^{-f}\left(1+\frac{1}{8f}\right)\left(1-\frac{1}{8\pi f}\right) \qquad \textbf{Eqn.2}$$

These cubics' coefficients were established by fitting EXCEL Trendlines to the plots of Quotient Residuals Γ/Γ_{approx} against Reciprocal Argument 1/z for 501 equispaced 1/z between chosen Lower and Upper Bounds of z. Hence we may identify z≡f and Γ_{approx} as the outcome of Equation One.

Standards of Comparison

The Gamma Function Equations W_0 and W_1 were compared with the fifth-degree Hastings Polynomial approximation and the eight-degree Hastings Polynomial as well as the Stirling Formula enhanced by a five-term series expansion (four terms excluding a unitary multiplier)[1].

The five functions were compared in terms of their RMS errors relative to a Γ intrinsic function of EXCEL, assumed fiducial.

The RMS (Root Mean Square) Error Function took the form:-

$$\varepsilon = \sqrt{\frac{\sum_{i=1}^{501}(\Gamma - \Gamma_{approx})^2}{501}} \qquad \textbf{Eqn.3}$$

All Hastings Polynomials take the form:-

$$^{n}H(z-1) = \sum_{i=1}^{m} C_i.(z-1)^i \qquad \textbf{Eqn.4}$$

where m may be 5 or 8 according to the precision required. The coefficients of ^5H and ^8H, respectively the five- and eight-degree forms, are tabulated below:-

Coefficient Subscript	^5H Coefficient	^8H Coefficient
0	1	1
1	-0.5748646	-0.577191652
2	0.9512363	0.988205891
3	-0.6998588	-0.897056937
4	0.4245549	0.918206857
5	-0.1010678	-0.756704078
6		0.482199394
7		-0.193527818
8		0.035868343

Table One
Hastings Estimator Coefficients

The third standard estimator for comparison was an extended Stirling Formula defined by:-

$$\Gamma_{approx} \approx e^{-z} \cdot z^{z-\frac{1}{2}} . \sqrt{2\pi} . \left(1 + \frac{1}{12z} + \frac{1}{288 z^2} - \frac{139}{51840 z^3} - \frac{571}{2488320 z^4} \right) \quad \textbf{Eqn.5}$$

The Cubic Corrector $^3P(f^{-1})$

Numerical Experiments employing EXCEL showed that a cubic equation was the optimal enhancer for W_0, superior to quadratics, exponential fitments, successive power-law fitments or other regressions accessible via Trendline.

However, factorisation of these cubics significantly vitiated accuracy, whilst no one cubic would offer comparability with Hastings accuracies for the broad range z=0 to 40. In fact, my estimators W_0 and W_1 are pretty useless in the range z=0 to 1.

Accordingly, I elected to compute the coefficients of $^3P(f^{-1})$ for the two ranges:-

A. z=1 to 2 (inclusive of a Γ/Γ_{approx} minimum)
B. z=1 to 12

the latter of which yields factorials into the hundreds of millions.

The regression coefficients for $^3P(f^{-1})$ for Cases A and B are tabulated below:-

Coefficient Degree	Range A	Range B
0	0.999798048389	0.999985402531
1	0.000707514046	0.001659904663
2	0.006136240729	0.007674764874
3	0.001339556824	0.002127685900

Table Two
Enhancer Polynomial Coefficients

These coefficients are employable such that:-

$$W_1 \approx {}^3P(f^{-1}).W_0 \approx W_0 \sum_{j=0}^{3} C_j . (f^{-1})^j \quad \textbf{Eqn.6}$$

The Relative Accuracy of The Five-Methods

The RMS Errors of the five methods of gamma function estimation are tabulated below for each z-Range:-

Method	Range A	Range B
W_0	0.00191309	557.37113
W_1	0.00000017428	11.3958148
^5H	0.000032634	5609104.1
^8H	0.00000015277	4665927.8
Stirling	0.00016952	0.01809828

Table Three
RMS Errors for Alternative Gamma Estimators

It is clearly seen that for the Key Range A (z=1 to 2), from which any gamma function can of course be replicated, the Enhanced Stirling Approximation W_1 contends with the eighth-degree Hastings Polynomial for accuracy. Even the extended Stirling Formula of Equation Five cannot match their precisions.

As may be expected, all precisions are much worse for the larger Range B. Hastings formulae break down completely. W_1 is clearly second-best, the extended Stirling Formula having come into its own as the more accurate method.

Recommendations

(1) Mill time studies in Range A should compare W_1 with ^8H.

(2) Mill time studies in Range B should compare W_1 and the extended Stirling Formula.

Reference

1 "Handbook of Mathematical Functions"
Edited by Milton Abramowitz and Irene A Stegun
Dover Publications of New York 1965
ISBN 0-486-61272-4

Notation

Γ	The Gamma Function
Γ_{approx}	an Approximation of a Gamma Function
$\Gamma(z)$	The Gamma Function of Argument z
C_i	The ith. Polynomial Coefficient
ε	The Root Mean Square Error
e	The Napierian Base
f	The Number whose Gamma Function is to be Found
${}^{m}H$	The m-Degree Hastings Polynomial
${}^{m}H(z-1)$	The m-Degree Hastings Polynomial for Argument (z-1)
π	The Ludolphine Constant
${}^{3}P(f^{-1})$	a Cubic Regression Polynomial for W_0 Enhancement
W_0	The Power-Law Enhanced Stirling Estimator
W_1	The Cubic and Power-Law Enhanced Stirling Estimator
z	The Number whose Gamma Function is to be Found