An Approximation of the Complete Gamma Function for Very Small Arguments

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This treatment is for arbitrary real arguments z.
For:-

$$z \le \frac{1}{2^{18}}$$

Equation 1

The absolute value of the Specific Defect, $|D_s|$, is:-

$$|D_s| = \left|\frac{\Gamma_{est} - \Gamma_{fid}}{\Gamma_{fid}}\right| < 10^{-14}$$

Equation 2

The method is based upon the Power Series for the Napierian Logarithm of Gamma[1] (Abramowitz and Stegun: 6.1.33):-

$$Log_n\Gamma = -Log_n(1+z) + z(1-\gamma) + \sum_{j=2}^{\infty}(-1)^j(\zeta_j - 1)\frac{z^j}{j}$$

Equation 3

And:-

$$\Gamma_s = e^{Log_n\Gamma}$$

Equation 4

It can be demonstrated that Equations Three and Four are sufficiently general to yield Gamma estimates conforming to error Equation Two for negative arguments of magnitude conforming to Equation One.

Since $0<\Re z<1$ we may utilise the Reflection Formula for Gamma:-

$$\Gamma_{est} = \frac{\pi}{\Gamma_s \times Sin(\pi z)}$$

Equation 5

that also shows fourteen-figure accuracy for positive arguments z.

Power Series Terms with Negative z

For –z in the defined range of very small absolute arguments we may resolve Equation Three for the Power Series as:-

$$Log_n\Gamma = T1 + T2 + T3$$

Equation 6

where:-

$$T1 = -Log_n\left[1 + (-z)\right]$$

Equation 7

$$T2 = -z(1-\gamma)$$

Equation 8

$$T3 = \sum_{j=2}^{2}(-1)^j\left(\zeta_j - 1\right)\frac{-z^j}{j}$$

Equation 9

Equation Nine is of course an approximation for the Third Term of the Power Series Equation.

Noting that:-

$$\zeta_2 = \frac{\pi^2}{6}$$

Equation 10

we may recast T3 as:-

$$T3 = \sum_{j=2}^{2}(-1)^j\left(\zeta_2 - 1\right)\frac{-z^j}{j} = \frac{-z^2}{2}\left(\frac{\pi^2}{6} - 1\right)$$

Equation 11

Furthermore:-

$$\left|-z\right| \approx T1 \approx -Log_n(1-z)$$

Equation 12

In addition to this we may demonstrate that, in the limit j→∞, the ratio T1/T2 gives:-

$$\frac{T1}{T2} = \frac{Log_n\left[1+(-z)\right]}{-z(1-\gamma)} = \sum_{j=1}^{\infty}\frac{1}{j(1-\gamma)}\cdot(-z)^{j-1} = \frac{1}{1-\gamma}$$

Equation 13

The Approximation of Gamma for Very Small Arguments

Noting that $-z^2 \equiv z^2$ we may substitute the T3 of Equation Eleven in the Power Series Equation Three to obtain:-

$$Log_n \Gamma(1-z) = -Log_n(1-z) - z(1-\gamma) + \frac{z^2}{2}\left(\frac{\pi^2}{6} - 1\right)$$

Equation 14

Now by the Reflection Formula we obtain:-

$$\Gamma(z) \approx \frac{\pi}{Sin(\pi z).Exp\left[-Log_n(1-z) - z(1-\gamma) + \frac{z^2}{2}\left(\frac{\pi^2}{6} - 1\right)\right]}$$

Equation 15

And noting that for $z \to 0$:-

$$Sin(\pi z) \approx \pi z$$

Equation 16

We may write:-

$$\Gamma(z) = \frac{\pi}{Sin(\pi z).e^{Log_n(\Gamma(1-z))}} = \frac{1}{z\Gamma(1-z)} \approx \frac{1}{z} - \gamma$$

Equation 17

that is an approximation of $\Gamma(z)$ for any real positive very small argument z.

Equation Seventeen may alternatively be expressed as:-

$$\Gamma(z) = \frac{\pi}{Sin(\pi z).e^{Log_n(1-z)}} \approx \frac{1}{z(1-z)} - (1+\gamma) = \frac{1}{z - z^2} - (1+\gamma)$$

Equation 18

Scratchpad Note

These analyses were validated with the assistance of the MATHCAD® computational scratchpad. The scratchpad file was stored as GAMTINY3.MCD of 20 December 2004.

MATHCAD® provided the fifteen-figure fiducial values of the Complete Gamma Function.

Reference

1 “Handbook of Mathematical Functions”
Edited by Milton Abramowitz and Irene A Stegun
Dover of New York 1965
ISBN 0-486-61272-4

Notation

D_s	The (Fractional) Specific Defect
γ	The Euler-Mascheroni Constant (A&S 6.1.3)
Γ	The Complete Gamma Function (CGF)
Γ_{est}	The Approximation of the CGF
Γ_{fid}	The Fiducial Value of the CGF
Γ_s	The CGF according to the Logarithm Power Series
π	The Ludolphine Constant
T_k	The kth. Power Series Term
ζ_j	The Reimann Zeta Function at Integer j
z	The Argument of the Complete Gamma Function