The Volumetry of the Bronshtein Obelisk and its implications for the Approximation of The Square Root

by
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Consider a solid hexahedron whose top and basal planes are parallel and whose opposite lateral and terminal planes' azimuths are parallel and have oppositely identical slopes. This figure suggests an old-fashioned ingot of cast gold or silver and is called by Bronshtein and Semendyayev[1] an "obelisk", though it conforms only to the frustal part of a Pharaonic obelisk: The part if such may be imagined as lacks its pyramidal finial.

Productions of the lateral and terminal planes do, in general, meet in a horizontal ridge, making a pentahedron suggestive of a housebrick frog or a hipped roof. Only in the case of an obelisk of square base and square top (one which is in fact the frustum of a square pyramid) will these planes meet at a point apex.

Figure One illustrates a perspective of such a volume viewed from a steep superior angle:-

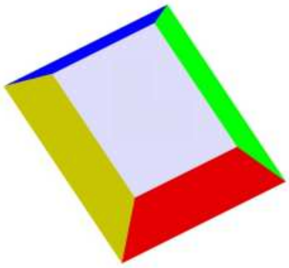

Figure 1

Given its bilateral symmetry in both vertical planes of intersection, it is possible to define any such obelisk with only five linear measures:-

Perpendicular Height	h
Breadth of Base	u_1
Length of Base	v_1
Breadth of Top	u_2
Length of Top	v_2

In addition it is convenient for our studies to define the auxiliary corner lengths a and b as:-

$$a = u_1 - u_2$$

Equation 1

and:-

$$b = v_1 - v_2$$

Equation 2

The auxiliary corner lengths may be, *but are not necessarily*, the same in value.

These measures of the obelisk are illustrated in Figure Two:-

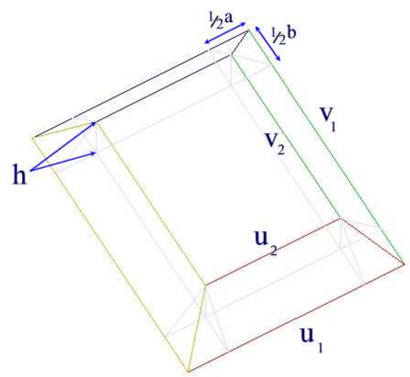

Figure Two

The Components of Obelisk Volume

As a matter of convenience we may resolve the component volumes of the obelisk to nine: One central cuboid of volume CCV; Two terminal prisms of right triangular section with a combined volume TPV; Two lateral prisms of right triangular section with a combined volume LPV; and Four corner skew pyramids with identical square or rectangular bases. These corner pyramids have a collective volume P.

Figure Three shows the central cuboid in light green:-

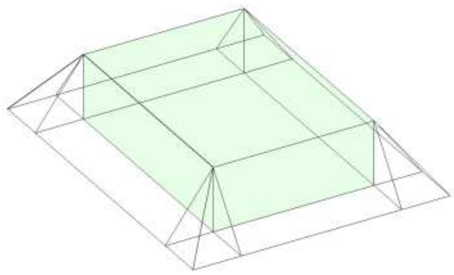

Figure 3

Figure Four shows the terminal prisms in light red:-

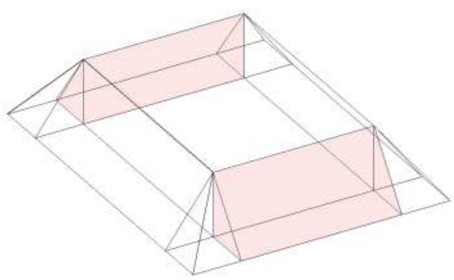

Figure 4

Figure Five shows the lateral prisms in light blue:-

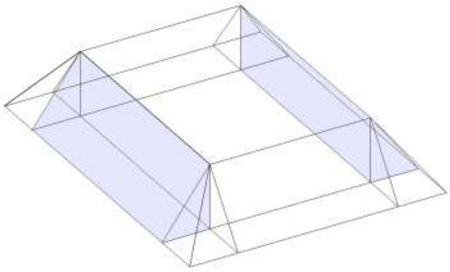

Figure 5

And Figure Six illustrates the corner skew pyramids respectively in yellow, blue, red and green:-

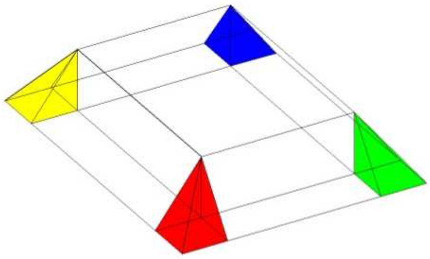

Figure 6

Before we review the algebra of the volumetrics it is convenient to introduce the Areas of the Top and Basal Planes of the Obelisk.

The Area of the Basal Plane, B_1, is given by:-

$$B_1 = u_1 v_1$$

Equation 3

and The Area of the Top Plane, B_2, is given by:-

$$B_2 = u_2 v_2$$

Equation 4

Accordingly, the Volume of the Obelisk, V_a, is given by:-

$$V_a = CCV + LPV + TPV + P$$

Equation 5

This can be recast as:-

$$V_a = hu_2v_2 + 2\times\frac{1}{2}\times\left[hv_2\left(\frac{1}{2}(u_1-u_2)\right)\right] + 2\times\frac{1}{2}\times\left[hu_2\left(\frac{1}{2}(v_1-v_2)\right)\right] + P$$
$$= hu_2v_2 + \frac{hv_2}{2}(u_1-u_2) + \frac{hu_2}{2}(v_1-v_2) + P$$
$$= \frac{h}{2}\left[2u_2v_2 + v_2(u_1-u_2) + u_2(v_1-v_2)\right] + P$$

Equation 6

Therefore:-

$$V_a = \frac{h}{2}\left[2B_2 + v_2(u_1-u_2) + u_2(v_1-v_2)\right] + P$$

Equation 7

The corner pyramids may conceptually be isolated by subtraction of the other volumes and, in virtue of their inner isohedrals, consolidated into one rectangular right-pyramid. This unification is illustrated in Figure Seven and the dimensions of the unitary pyramid shown in Figure Eight.

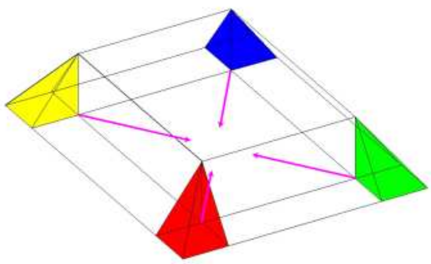

Figure 7

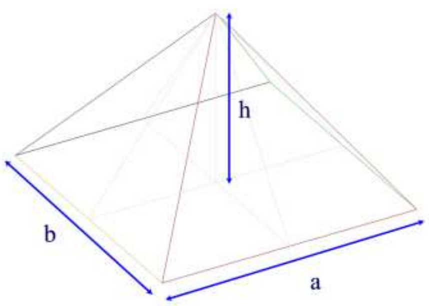

Figure 8

It can be shown that the Volume of the Unitary Pyramid, P, is given by:-

$$P = \frac{1}{3}abh$$

Equation 8

which may be re-expressed as:-

$$P = \frac{h}{3}(u_1 - u_2)(v_1 - v_2)$$

Equation 9

Therefore Equation Six may be recast as:-

$$V_a = hu_2v_2 + 2\times\frac{1}{2}\times\left[hv_2\left(\frac{1}{2}(u_1 - u_2)\right)\right] + 2\times\frac{1}{2}\times\left[hu_2\left(\frac{1}{2}(v_1 - v_2)\right)\right] + 2\times\frac{1}{6}(u_1 - u_2)(v_1 - v_2)h$$

Equation 10

Equation Ten and Equation Seven have many identities. An elegant selection from Bronshtein includes:-

$$V_a = \frac{h[(2v_1 + v_2)u_1 + (2v_2 + v_1)u_2]}{6}$$

Equation 11

$$V_a = \frac{h}{6}[B_1 + (v_1 + v_2)(u_1 + u_2) + B_2]$$

Equation 12

$$V_a = \frac{h}{6}[2B_1 + (u_1v_2 + v_1u_2) + 2B_2]$$

Equation 13

For our immediate purposes, however, we may simplify Equation Ten to:-

$$V_a = \frac{1}{3}hu_2v_2 + \frac{1}{6}hv_2u_1 + \frac{1}{6}hu_2v_1 + \frac{1}{3}hu_1v_1$$

Equation 14

Which by collection of the parallel planes gives:-

$$V_a = \frac{h}{3}(B_1 + B_2) + \frac{h}{6}(v_2u_1 + u_2v_1)$$

Equation 15

The final convenient expression for the Volume of a Bronshtein obelisk may harmonise Equation Fifteen in h/6 to yield:-

$$V_a = \frac{h}{6}[v_2u_1 + u_2v_1 + 2(B_1 + B_2)]$$

Equation 16

The Frustum of a Square Pyramid

The frustum (or "frustrum") of a square pyramid is a special case of a Bronshtein obelisk in which $u_1=v_1$ and $u_2=v_2$.

The Volume of a Square Pyramid Frustum, V_b, is given exactly by[2]:-

$$V_b = \frac{h}{3}\left(B_1 + \sqrt{B_1 B_2} + B_2\right)$$

Equation 17

and approximately by:-

$$V_b = \frac{h(B_1 + B_2)}{2}$$

Equation 18

Now it is notable that the volumes of a variety of anisometric Bronshtein obelisks are well-approximated by those of square-pyramid frusta of Equivalent B_1 and B_2.

Therefore we may write:-

$$\frac{h}{3}\left(B_1 + B_2 + \sqrt{B_1 B_2}\right) \approx \frac{h}{6}\left[v_2 u_1 + u_2 v_1 + 2(B_1 + B_2)\right]$$

Equation 19

By elimination of Height h from each side, and doubling:-

$$\frac{2}{3}\left(B_1 + B_2 + \sqrt{B_1 B_2}\right) \approx \frac{1}{3}\left[v_2 u_1 + u_2 v_1 + 2(B_1 + B_2)\right]$$

Equation 20

By collecting two-thirds the sums of the top and basal areas:-

$$\frac{2}{3}\left[B_1 + B_2\right] + \frac{2}{3}\sqrt{B_1 B_2} \approx \frac{1}{3}\left(u_2 v_1 + v_2 u_1\right) + \frac{2}{3}\left[B_1 + B_2\right]$$

Equation 21

Accordingly:-

$$\frac{2}{3}\sqrt{B_1 B_2} \approx \frac{1}{3}\left(u_2 v_1 + v_2 u_1\right)$$

Equation 22

Multiplying both sides by 3/2:-

$$\sqrt{B_1 B_2} \approx \frac{1}{2}\left(u_2 v_1 + v_2 u_1\right)$$

Equation 23

It is therefore the case that the square root may be approximated as a simple function of four factors.

The quality of this estimate depends upon the magnitude of the square and the choice of its four factors.

Example

Compute the approximate square-root of 120 and express its precision in terms of percentage specific defect.

120 may be factored as 2×3×4×5.

Accordingly:-

$u_1=4 \quad v_1=5 \quad u_2=2 \quad v_2=3$

The approximate square-root is given by:-

$$\begin{aligned}\sqrt{120} &\approx \frac{1}{2}(2\times5+3\times4)\\ &\approx \frac{1}{2}(10+12)\\ &\approx 11\end{aligned}$$

Now the precise square root to ten decimal places is 10.9544511501.

Therefore the percentage specific defect is:-

$$d_s\% = 100\left(\frac{estimate - exact}{exact}\right) = 100\left(\frac{11-10.9544511501}{10.9544511501}\right) = 0.41580220928$$

There is accordingly and error of about +0.4158% in the approximation of the square root of 120 by this method.

References

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