An Approximate Square Root by means of the Partition of the Obelisk

by
James R Warren BSc MSc PhD PGCE

Consider a Bronshtein Obelisk[1], a hexahedron of the kind illustrated by the perspective view in Figure One:-

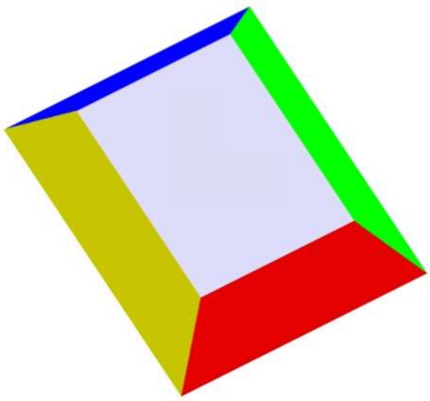

Figure 1

The volumetry of such an elongated frustum differs from that of a square frustum, though the analytic algebra is similar, and numerical results are often nearly equivalent.

Notwithstanding the fact that an obelisk's opposing slope pairs (glacises) may differ in inclination, we can divide the plenary obelisk into what is approximately a square frustum, plus some complementary volume.

This completing volume can be some (usually) smaller obelisk, if we will allow that to interpenetrate the sub-frustum.

A third component volume includes the volumetric overlap between the sub-frustum and the residual obelisk. This volume constitutes a Bronshtein wedge, that for convenience we shall style a frog, because its geometry is that of the pentahedral open depression that English brickmakers impress in their product.

Figure Two shows the way in which the sub-frustum fits into the original plenary obelisk:-

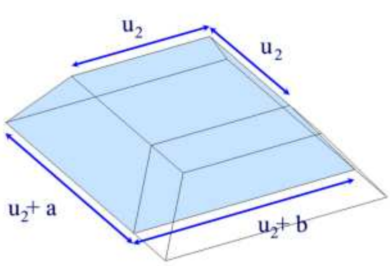

Figure 2

Figure Three illustrates the minor obelisk that occupies the other end of the plenary obelisk:-

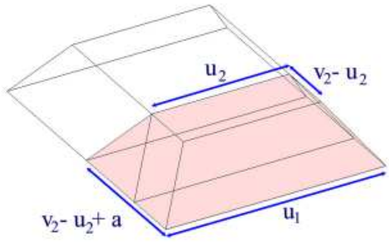

Figure 3

Figure Four shows the frog that overlaps the two component obelisks.

A frog is really an upward production of the sides of an obelisk until they meet in a horizontal ridge.

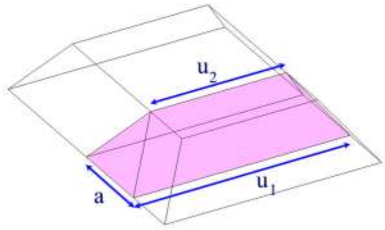

Figure 4

Derived Dimensions

Convenient derived glacis lengths may be defined as:-

$$a = v_1 - v_2$$

Equation 1

$$b = u_1 - u_2$$

Equation 2

whilst the Areas of the Plenary Obelisk's Basal and Top Planes are respectively:-

$$B_1 = u_1 v_1$$

Equation 3

$$B_2 = u_2 v_2$$

Equation 4

Polyhedron Volumes

Strictly, the plenary solid and its components are all obelisks.
But for an exact Square Pyramidal Frustum:-

$$V_f = \frac{h}{3}\left(B_1 + B_2 + \sqrt{B_1 B_2}\right)$$

Equation 5

where V_f is the Frustum Volume and h is the Obelisk Perpendicular Height.

For a general Obelisk, the Volume V_k is given by:-

$$V_k = \frac{h}{6}\left[2B_1 + (u_1 v_2 + v_1 u_2) + 2B_2\right]$$

Equation 6

Square Pyramidal Frusta and general Obelisks are hexahedra. A frog is a pentahedron.

The Volume of a Frog, V_g, is given by:-

$$V_g = \frac{1}{6} ha(2u_1 + u_2)$$

Equation 7

The Approximation of the Sub-Frustum frm1

V_{frm1}, the Volume of the Major Component Obelisk, the Sub-Frustum frm1 is computed exactly using Equation Six.

But we have arranged that V_{frm1} is quite or very nearly the volume of the pyramidal frustum given by Equation Five.

Accordingly:-

$$V_{frm1} \approx V_f$$

Equation 8

This is done by setting the top length of the solid to its top breadth for a square top plane, conserving glacis dips in the process.

Therefore, with appropriate substitutions:-

$$V_{frm1} \approx \frac{1}{3} h\left[(u_2 + a)(u_2 + b) + u_2{}^2 + \sqrt{(u_2 + a)(u_2 + b)u_2{}^2}\right]$$

Equation 9

which may be condensed to:-

$$V_{frm1} \approx \frac{u_2 h}{3}\left(2u_2 + b + a + \frac{ab}{u_2} + \sqrt{u_2 + a}.\sqrt{u_2 + b}\right)$$

Equation 10

The Exact Computation of the Minor Obelisk frm2

Reference to the solid lengths and Equation Six enables us to formulate the Volume of frm2 as:-

$$V_{frm2} = \frac{h}{6}\{2u_1[(v_2 - u_2) + a] + u_2[(v_2 - u_2) + a] + u_1(v_2 - u_2) + 2u_2(v_2 - u_2)\}$$

Equation 11

that can be condensed to:-

$$V_{frm2} = \frac{1}{6}h\left[3u_1 v_2 - 3u_1 u_2 + 2u_1 a - 3u_2 v_2 - 3u_2{}^2 + au_2\right]$$

Equation 12

During computations, V_{frm2} is capable of negative values without prejudice to the approximation of square roots.

The Exact Computation of the Frog

The Frog Volume, V_g, is given by:-

$$V_g = \frac{h}{6}(2u_1 a + u_2 a + u_1 \times 0 + 2u_2 \times 0)$$

Equation 13

which simplifies to:-

$$V_g = \frac{ha}{6}(2u_1 + u_2)$$

Equation 14

The Exact Computation of the Plenary Obelisk

The Plenary Obelisk Volume, V_k, is given by:-

$$V_k = \frac{h}{6}[2u_1 v_1 + (u_1 v_2 + v_1 u_2) + 2u_2 v_2]$$

Equation 15

The Approximation of the Plenary Obelisk

By setting Equation Fifteen to the sum of the component obelisk volumes minus the frog volume we can approximate the plenary obelisk on our way to the approximation of square roots:-

$$V_k \approx V_{frm1} + V_{frm2} - V_g \approx V_{comp}$$

Equation 16

The Composite Volume, V_{comp}, is, by in-substitution of Equations Ten, Twelve and Fourteen:-

$$V_{comp} \approx \frac{u_2 h}{3}\left(2u_2 + b + a + \frac{ab}{u_2} + \sqrt{u_2 + a}.\sqrt{u_2 + b}\right)$$
$$+ \frac{1}{6}h\left[3u_1v_2 - 3u_1u_2 + 2u_1a + 3u_2v_2 - 3u_2^{\ 2} + au_2\right] + \frac{ha}{6}(2u_1 + u_2)$$

Equation 17

which may be reduced to:-

$$V_{comp} \approx \frac{h}{6}\left[-u_2^{\ 2} - u_1u_2 + 3v_2u_2 + 2v_1u_1 + u_1v_2 + 2\sqrt{u_2 + v_1 - v_2}.\sqrt{u_1}.u_2\right]$$

Equation 18

Setting this approximately equal to the Plenary Volume of Equation Fifteen we obtain:-

$$\frac{h}{6}\left[2u_1v_1 + (u_1v_2 + v_1u_2) + 2u_2v_2\right] = \frac{h}{6}\left[-u_2^{\ 2} - u_1u_2 + 3v_2u_2 + 2v_1u_1 + u_1v_2 + 2\sqrt{u_2 + v_1 - v_2}.\sqrt{u_1}.u_2\right]$$

Equation 19

Elimination of h/6 followed by re-arrangement and simplification enables us to write:-

$$v_1u_2 - v_2u_2 + u_2^{\ 2} + u_1u_2 \approx 2\sqrt{u_2 + v_1 - v_2}.\sqrt{u_1}.u_2$$

Equation 20

That can be further reduced to:-

$$\frac{1}{2}(u_1 + u_2 + v_1 - v_2) \approx \sqrt{u_1}.\sqrt{u_2 + v_1 - v_2}$$

Equation 21

Noting that $a = v_1 - v_2$ we may reduce Equation Twenty-One to:-

$$\frac{u_1 + u_2 + a}{2} \approx \sqrt{u_1}.\sqrt{u_2 + a}$$

Equation 22

Therefore:-

$$\sqrt{u_1} \approx \frac{u_1 + u_2 + a}{2\sqrt{u_2 + a}}$$

Equation 23

Finally, if we allow that:-

$$w = u_2 + a$$

Equation 24

then:-

$$\sqrt{u_1} \approx \frac{u_1 + w}{2\sqrt{w}}$$

Equation 25

In practice, the approximate root of u_1 is determinable by fixing w as some convenient square of an integer just greater than u_2. Meanwhile, u_2 can be a convenient integer around u_1 or even the (real) u_1 itself. In this context, a is thus the difference between the integer square and the convenient u_2.

The quality of the roots approximated by Equation Twenty-Five is uneven, and when u_1 is itself an integer root there can be serious error. There is some tentative suggestion that the greater is a, the better the estimate.

<u>Example</u>

Find the square root of u_1 = 4687.91487.
We will approximate the root assuming:-

(a)	u_2 = 4600	u_2+a = 5625
(b)	u_2 = 4600	u_2+a = 4761
(c)	u_2 = 5000	u_2+a = 10000

Fiducial $u_1^{0.5}$ = 68.4683494032

$$\sqrt{u_1} \approx \frac{u_1 + w}{2\sqrt{w}} \qquad w \approx u_2 + a \qquad d_s\% = 100\left(\frac{estimate - exact}{exact}\right)$$

(a) $$\sqrt{u_1} \approx \frac{4687.91487 + 5625}{2\sqrt{5625}} = 68.7527658$$

$$d_s\% = 0.415398354$$

(b) $$\sqrt{u_1} \approx \frac{4687.91487 + 4761}{2\sqrt{4761}} = 68.4703976087$$

$$d_s\% = 0.00299146321$$

(c) $$\sqrt{u_1} \approx \frac{4687.91487 + 10000}{2\sqrt{10000}} = 73.43957435$$

$$d_s\% = 7.26061748256$$

In most situations the choice of w is the controlling factor for accuracy, and the closer it approaches u_1 the better.

To illustrate this we will choose a w value at 4688:-

(d) $$\sqrt{u_1} \approx \frac{4687.91487 + 4688}{2 \times 68.4689710745} = 68.468349406$$

$$d_s\% = 4.12 \times 10^{-9}$$

Some Statistical Observations upon Root Estimate Quality

Table One summarises the Mean, $\mu[d_s\%]$, Population Standard Deviation, $\sigma[d_s\%]$, and Percentage Coefficient of Variation, $CV[d_s\%]$, of the Specific Defects of one hundred square root estimates of random real numbers in the nine ranges specified.

In the course of fifteen independent trials it was computed that:-

$$\mu[d_s\%] = 68.28859793\,p^{-1.02898190} \approx \frac{68.3}{p}$$

Equation 26

where p is the Range Mid-Point. The Coefficient of Determination r^2 was 0.86673344.

The relationship between the Mean Specific Defect and the Range Mid-Point is shown in Figure Five:-

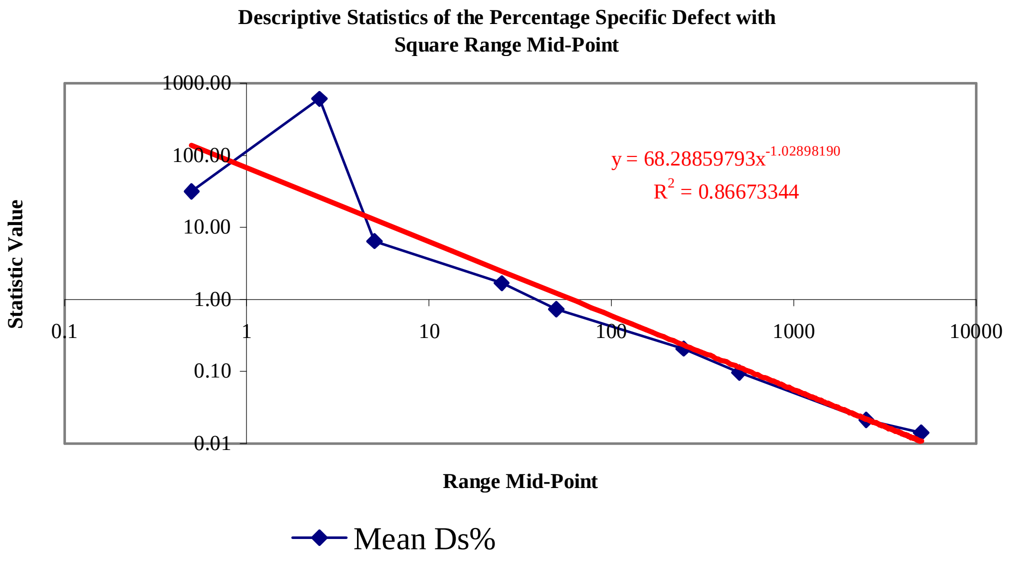

Figure 5

For the same fifteen trials:-

$$\sigma[d_s\%] = 21.53809447\,p^{-0.94401713} \approx \frac{21.5}{p}$$

Equation 27

$r^2 = 0.84088179$.

The relationship between the Population Standard Deviation of the Specific Defects and the Range Mid-Point is shown in Figure Six:-

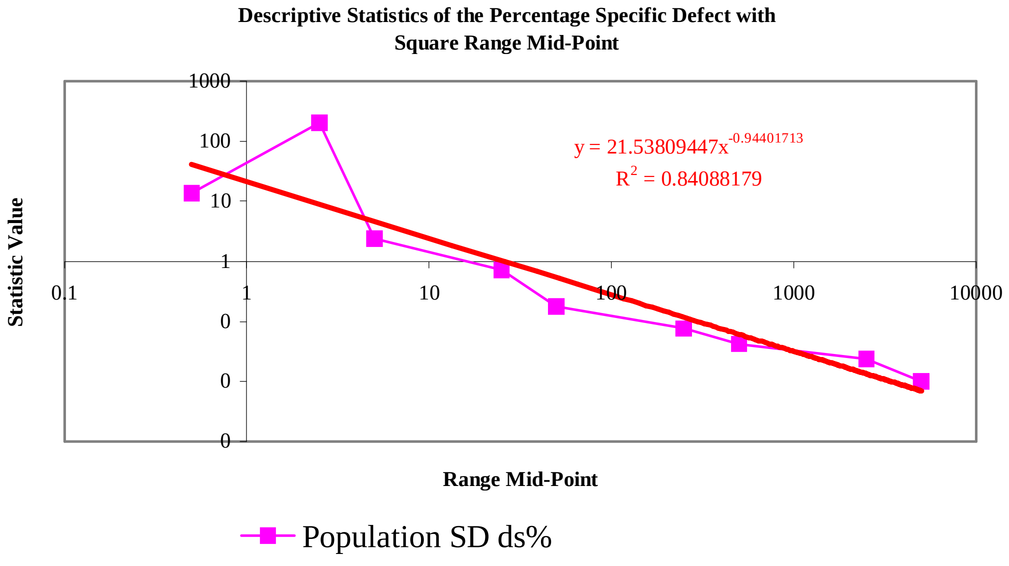

Figure Six

Lastly:-

$$CV[d_s\%] = 0.01084671p + 39.36518708 = 40 + 0.01p$$

Equation 28

$r^2 = 0.49246535$.

The quasi-linear relationship between the Specific Defects' Coefficient of Variation and the Range Mid-Point is shown in Figure Seven (note the linearity of the red regression line in the loglog scaling of the graph):-

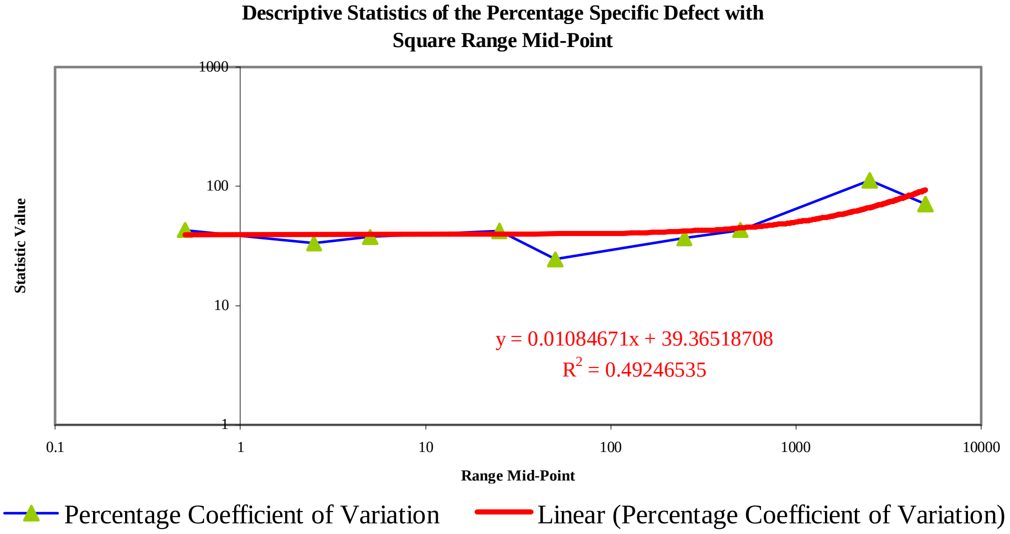

Figure 7

Matrix of d_s%

Trial 1	38.100542	480.780833	5.934170	1.554613	1.030332	0.124122	0.078288	0.015098	0.005490
Trial 2	24.181812	571.320851	9.944211	0.944259	0.728789	0.270687	0.086219	0.021418	0.010885
Trial 3	22.513360	426.695219	5.275247	2.102492	0.826411	0.166578	0.082068	0.011058	0.011574
Trial 4	26.238208	713.846767	10.293007	1.185231	1.055782	0.138226	0.088355	0.022812	0.041197
Trial 5	17.793009	822.963458	4.223849	1.427051	0.474207	0.224560	0.141344	0.010813	0.012601
Trial 6	16.541293	474.564340	9.425815	1.013810	0.551288	0.384511	0.097333	0.017709	0.029401
Trial 7	36.555915	639.905688	3.909584	1.584159	0.522660	0.166408	0.131249	0.010425	0.012550
Trial 8	67.613794	1219.721130	9.806852	1.259928	0.441711	0.220751	0.085217	0.011984	0.008016
Trial 9	58.160208	366.432520	6.286489	1.512468	0.694862	0.131448	0.085858	0.010837	0.005854
Trial 10	30.712649	546.518190	3.965775	1.681368	0.848771	0.075024	0.097603	0.020118	0.004327
Trial 11	27.026255	552.154137	4.960024	1.502007	0.682134	0.309950	0.085445	0.010028	0.008918
Trial 12	25.330721	617.066255	3.523097	3.947829	0.807520	0.217239	0.042683	0.016055	0.027532
Trial 13	27.060746	440.024941	5.951021	2.493676	0.725094	0.196256	0.058597	0.108388	0.007792
Trial 14	25.984880	716.467147	8.839849	1.809229	0.852823	0.228771	0.067041	0.014176	0.015630
Trial 15	28.214800	502.797998	4.071321	1.317437	0.674841	0.256329	0.225594	0.016130	0.010228
Range Lower Bound	0	0	0	0	0	0	0	0	0
Range Upper Bound	1	5	10	50	100	500	1000	5000	10000
Range Mid-Point	0.5	2.5	5	25	50	250	500	2500	5000
Mean D_s%	31.468546	606.083965	6.427354	1.689037	0.727815	0.207391	0.096860	0.021137	0.014133
Population SD d_s%	13.607215	202.841329	2.432467	0.715801	0.177167	0.076504	0.041761	0.023665	0.010095
Percentage Coefficient of Variation	43.240684	33.467529	37.845537	42.379232	24.342267	36.888994	43.114573	111.961822	71.431697

Table 1
Solutions Statistics

Notation

a	The Auxiliary Corner Length = $v_1 - v_2$ (The Root Completion Additive)
b	The Auxiliary Corner Breadth = $u_1 - u_2$
B_1	The Basal Area of the Plenary Obelisk
B_2	The Top Area of the Plenary Obelisk
CV	The Coefficient of Variation
d_s%	The Percentage Specific Defect
h	The Obelisk Perpendicular Height
μ	The Arithmetic Mean
p	The Range Mid-Point (of square-rooted numbers u_1)
r^2	The Coefficient of Determination
σ	The Population Standard Deviation
u_1	The Breadth of the Base
u_2	The Breadth of the Top
v_1	The Length of the Base
v_2	The Length of the Top
V_{comp}	The Approximate Volume of the Plenary Obelisk
V_{frm1}	The Approximate Volume of the Sub-Frustum
V_{frm2}	The Exact Volume of the Minor Obelisk
V_g	The Exact Frog Volume
V_k	The Exact Volume of the Plenary Obelisk
w	The Subject Square = u_2 + a

Reference

1 “Handbook of Mathematics”
IN Bronshtein and KA Semendyayev
Translated by KA Hirsch
Van Nostrand Reinhold and Company of New York 1985
ISBN 0-442-21171-6
pp973
Page 179

25 April 2005