Some Findings in the Statistical Estimation Of the Area and Perimeter Of the General Polygon

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This study develops the use of momental statistics of polygonal radii in the approximation of both the Area and the Perimeter. I explored the use of the mean and the (population) standard deviation on invariant radii and uniformly-distributed radii.

An economic motivation, the minimisation of both trigonometric solutions and of roots, has guided these elaborations.

Much work remains to be done in these areas:-

(a) Distributional Definition
(b) Metric Definition
(c) Reflexively-Bounded Shapes

Spreadsheet experiments and tabulations used EXCEL software.

PART I

ESTIMATION OF THE AREAS AND PERIMETERS OF REGULAR POLYGONS

In this preliminary study, polygonal radii were set to a constant (arbitrarily ten) and the angular error was set to zero percent. Accordingly a series of regular polygons were generated in which by definition Any Metric of Central Tendency, m, was $r_1=r_2=r_3=10$ and Any Metric of Dispersion s was zero.

Therefore, the Exact Area, A, was given by:-

$$A = \frac{\pi r^2}{2}.\text{Sin}\left(\frac{2\pi}{\text{n}}\right) \qquad \textbf{Eqn.1}$$

and the Exact Perimeter by:-

$$P = n.2r\ \text{Sin}\left(\frac{\pi}{\text{n}}\right) \qquad \textbf{Eqn.2}$$

Because of the equivalency of:-

$$nr^2 \equiv \sum_{i=1}^{n} r_i\, r_{i+1} \qquad \textbf{Eqn.3}$$

Under these conditions, A and its Summative Estimators of Part II collapsed to equivalence.

Similarly, P and its Part II estimators proved equivalent because:-

$$2\,\mathrm{Sin}\left(\frac{\pi}{n}\right) = \sqrt{2\left(1 - \mathrm{Cos}\left(\frac{2\pi}{n}\right)\right)} \qquad \textbf{Eqn.4}$$

PART II

THE ESTIMATION OF AREAS AND PERIMETERS USING CORRECTED APPROXIMATIONS

For convenience permit that the series of n polar co-ordinates $[\theta_i, r_i]$ define the bounding vertices of a general polygon.

Then:-

$$A = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{1}{2} r_i r_{i+1} \mathrm{Sin}\,\theta_i \qquad \textbf{Eqn.5}$$

where it is understood that $r_1 \equiv r_{n+1}$.

Further:-

$$P = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \sqrt{r_i^2 + r_{i+1}^2 - 2 r_i r_{i+1} \mathrm{Cos}\,\theta_i} \qquad \textbf{Eqn.6}$$

Now for large n, the InterAngles θ_i approximate zero and the following trigonometric approximations apply:-

$$\mathrm{Sin}\,\theta \approx \theta \qquad \textbf{Eqn.7}$$

Accordingly the estimation of Area and Perimeter can be economised by the substitution of the trigonometric functions.

It is, however, desirable to recover some of the accuracy thereby lost through the use of Corrector Functions. Firstly, the Areal Corrector, C_A, may be defined as:-

$$C_A = \frac{n\,\mathrm{Sin}\left(\frac{2\pi}{n}\right)}{2\pi} = \frac{n}{2\pi}\mathrm{Sin}\left(\frac{2\pi}{n}\right) \qquad \textbf{Eqn.9}$$

And secondly the Perimeter Corrector, C_P, is:-

$$C_P = \frac{\sqrt{2\left(1 - \mathrm{Cos}\left(\frac{2\pi}{n}\right)\right)}}{\frac{2\pi}{n}} = \frac{n}{2\pi}\sqrt{2\left(1 - \mathrm{Cos}\left(\frac{2\pi}{n}\right)\right)} \qquad \textbf{Eqn.10}$$

The Economised Polygonal Area, A_e, is then assessed using:-

$$A_e \approx \frac{C_A \pi}{n}\sum_{i=1}^{n} r_i r_{i+1} \qquad \textbf{Eqn.11}$$

whilst the Economised Polygonal Perimeter, P_e, is approximated by:-

$$P_e \approx C_P \cdot \sum_{i=1}^{n} \sqrt{r_1^2 + r_{i+1}^2 - 2 r_i r_{i+1}\left(1 - \frac{2\pi^2}{n^2}\right)} \qquad \textbf{Eqn.12}$$

The Corrected Approximations were tested for six non-reflexive polygons whose radii were uniformly-distributed in the range 5±2.5 units. Angular errors were twenty percent. Error Percent was defined as 100((estimate-exact value)/exact value).

Table One summarises the results showing the negligible effects of abandoning repeated trigonometric function solutions, especially in perimetric estimation.

It must be noted that the method fails for reflex-boundary areas which show multiple boundary crossings of radii.

ERROR ESTIMATES FOR
CORRECTED AREAL AND PERIMETRIC SUMMATIONS
OF RANDOM POLYGONS

Radius = 5						
Radial Error = 2.5						
Angular Error = 20%						
Number of Sides	4	8	16	32	64	360
Exact Area	35.8333	51.6129	88.6654	65.2289	85.1357	75.2874
Estimated Area	36.9377	52.3922	89.0771	65.9522	85.0518	75.5533
Exact Perimeter	26.3005	31.2989	45.6588	73.7465	91.3571	635.5672
Estimated Perimeter	25.9013	31.2099	45.1300	73.8402	91.1112	635.5513
Area Error Percent	2.9899	1.4874	0.4622	1.0967	-0.0986	0.3519
Perimeter Error Percent	-1.5178	-0.2844	-1.1582	0.1271	-0.2692	-0.0025

Table One

PART III

THE ESTIMATION OF PERIMETERS
USING THE MEAN TRANSITION

Any method which employs Carnot's Theorem to measure the boundary of a polygon is inherently expensive because it is a sum of square roots. If square roots can be minimised or eliminated savings can be made.

The proposal invokes a concept I call The Mean Transition (in length between two adjacent radii of a polygon whose vertices are defined by polar co-ordinates).

Perimeters can satisfactorily be estimated by this method if:-

(a) The Planar Closed Area's Boundary Cuts Each Radius Once Only (The Figure is Non-Reflexive)
(b) The Distribution of Radii is Uniform
(c) The Distribution of Radius InterAngles is Uniform
(d) The Number of Boundary Vertices is Large

Hence the method is most satisfactory for echinoidal splashes. Even so, estimates of the perimeter better than within 8% of Carnot values cannot be guaranteed.

The Basis of Definition

Let us commence with Carnot's Theorem applied to any particular triangular sector bounded by radii r_i and r_{i+1} including angle θ_i.

Then the Length of the Perimetric Element, p, which closes the sector is given by:-

$$p = \sqrt{r_i^2 + r_{i+1}^2 - 2\, r_i\, r_{i+1} \operatorname{Cos}\theta_i} \qquad \textbf{Eqn.13}$$

Dividing all terms by $r_i \times r_{i+1}$ gives:-

$$p = \sqrt{\frac{r_i}{r_{i+1}} + \frac{r_{i+1}}{r_i} - 2\operatorname{Cos}\theta_i} \qquad \textbf{Eqn.14}$$

Allow that as:-

$$\theta_i \to \theta \,:\, \operatorname{Cos}\theta_i \to \left(1 - \frac{\theta_i^2}{2}\right) \qquad \textbf{Eqn.15}$$

Then:-

$$p = \sqrt{\frac{r_i}{r_{i+1}} + \frac{r_{i+1}}{r_i} - 2\left(1 - \frac{\theta_i^2}{2}\right)} \qquad \textbf{Eqn.16}$$

Thirdly, for a sufficiently large Number of Polygon Sides (or Radial Vertices) n:-

$$\theta_i \approx \frac{2\pi}{n} \qquad \textbf{Eqn.17}$$

Hence:-

$$p \approx \sqrt{\frac{r_i}{r_{i+1}} + \frac{r_{i+1}}{r_i} - 2 + \frac{2\pi^2}{n^2}} \qquad \textbf{Eqn.18}$$

or:-

$$p \approx \sqrt{T + k} \qquad \textbf{Eqn.19}$$

where T is a Radial Length Transition Parameter and k is an Estimation Constant defined as:-

$$k = 2\left(\frac{\pi^2}{n^2} - 1\right) \qquad \textbf{Eqn.20}$$

The Mean Transition, μ_T

If the perimeter of a planar surface is to be estimated using metrics of the central tendency and dispersion of the set of approximating radii, then the characteristic average of the Transition Parameter, T, needs to be developed.

We may note that this Transition Parameter may be defined as:-

$$T = \sqrt{\frac{a}{b} + \frac{b}{a}} = \sqrt{t} \qquad \textbf{Eqn.21}$$

where a and b are respective statistical functions of the radii, r_i and r_{i+1} which can be employed to typify the transitions r_i to r_{i+1} and r_{i+1} to r_i.

Let us postulate that a and b can assume any of the three values m-s, m or m+s where m and s are respectively appropriate metrics of central tendency and dispersion.

Further, code 0=m-s; 1=m; 2=m+s so that the conjoint fractions of t constitute a ternary system in which the numerator of the first fraction has the highest place value 3^3=9; the articulated denominator of the first with numerator of the second has the middle place 3^1=3; and the denominator of the second fraction has place value 3^0=1.

Then it follows that any particular configuration of m-s, m and m+s allowable by the definition is a member of the 27 combinations elaborated upon Table Two.

For example:-

$$t[002] = \frac{m - s}{m - s} + \frac{m - s}{m + s} \qquad \textbf{Eqn.22}$$

whilst:-

$$t[101] = \frac{m}{m - 1} + \frac{m - 1}{m} \qquad \textbf{Eqn.23}$$

Now it is clear from the definition of t that only the nine symmetric configurations are of interest to us and these are emboldened in Table Two and segregated into Table Three for convenience.

It is nextly the case that we have to acquire the Total of the nine allowable t's as we proceed to the Mean Transition. Accordingly:-

$$\tau_t = \sum_{i=1}^{9} t_i =$$

$$\frac{m-s}{m-s} + \frac{m-s}{m-s} + \frac{m-s}{m} + \frac{m}{m-s} + \frac{m-s}{m+s} + \frac{m+s}{m-s}$$
$$+ \frac{m}{m-s} + \frac{m-s}{m} + \frac{m}{m} + \frac{m}{m} + \frac{m}{m+s} + \frac{m+s}{m}$$
$$+ \frac{m+s}{m-s} + \frac{m-s}{m+s} + \frac{m+s}{m} + \frac{m}{m+s} + \frac{m+s}{m+s} + \frac{m+s}{m+s} \qquad \textbf{Eqn.24}$$

where τ_t is the Total of t's.
By gathering fractions we summarise this elaboration as:-

$$\tau_t = \frac{2(m-s) + 2m + 2(m+s)}{m-s} + \frac{2(m-s) + 2m + 2(m-s)}{m} + \frac{2(m-s) + 2m + 2(m+s)}{m+s} \qquad \textbf{Eqn.25}$$

or:-

$$\tau_t = \frac{6m}{m-s} + \frac{6m}{m} + \frac{6m}{m+s} \qquad \textbf{Eqn.26}$$

for which further simplification yields:-

$$\tau_t = \frac{6m}{m-s} + 6 + \frac{6m}{m+s}$$
$$= 6 + 6m\left(\frac{1}{m-s} + \frac{1}{m+s}\right) \qquad \textbf{Eqn.27}$$

Each term should now be divided by its probabilistic loading (1/9) to establish the mean of t:-

$$\mu_t = \frac{6}{9} + \frac{6m}{9}\left(\frac{1}{m-s} + \frac{1}{m+s}\right)$$
$$\therefore \mu_t = \frac{2}{3} + \frac{2m}{3}\left(\frac{1}{m-s} + \frac{1}{m+s}\right) \qquad \textbf{Eqn.28}$$

This expression in turn simplifies to:-

$$\mu_t = \frac{2}{3}\left[1 + m\left(\frac{1}{m-s} + \frac{1}{m+s}\right)\right] \qquad \textbf{Eqn.29}$$

The Extended Transition Parameter, T_X

The Extended Transition Parameter, T_X, may be defined as:-

$$T_X = \sqrt{\mu_t + k} \approx \sqrt{\left(\frac{a}{b} + \frac{b}{a}\right) + k} \qquad \textbf{Eqn.30}$$

where:-

$$k = \frac{2\pi^2}{n^2} - 2 \qquad \textbf{Eqn.31}$$

and n is the Number of Vertices on the polygon periphery.
Hence:-

$$\begin{aligned} T_X &= \sqrt{\frac{2}{3}\left[1 + m\left(\frac{1}{m-s} + \frac{1}{m+s}\right)\right] + \frac{2\pi^2}{n^2} - 2} \\ &= \sqrt{\frac{2m}{3}\left(\frac{1}{m-s} + \frac{1}{m+s}\right) - \frac{4}{3} + \frac{2\pi^2}{n^2}} \end{aligned} \qquad \textbf{Eqn.32}$$

The Employment of T_X as a Perimetric Estimator

T_X is analytically an Angle which sweeps the Characteristic Radius, r_c, through an arc which generates the Estimated Perimeter, P_{est}. This radius is very long and is in fact the product of the Mean Radius, μ_r, and the Number of Radii, n:-

$$r_c = n\,\mu_r \qquad \textbf{Eqn.33}$$

Therefore:-

$$P_{est} = n\,\mu_r T_X \qquad \textbf{Eqn.34}$$

Coding Tables

Denary	Ternary		
	9	3	1
0	**0**	**0**	**0**
1	0	0	1
2	0	0	2
3	**0**	**1**	**0**
4	0	1	1
5	0	1	2
6	**0**	**2**	**0**
7	0	2	1
8	0	2	2
9	1	0	0
10	**1**	**0**	**1**
11	1	0	2
12	1	1	0
13	**1**	**1**	**1**
14	1	1	2
15	1	2	0
16	**1**	**2**	**1**
17	1	2	2
18	2	0	0
19	2	0	1
20	**2**	**0**	**2**
21	2	1	0
22	2	1	1
23	**2**	**1**	**2**
24	2	2	0
25	2	2	1
26	**2**	**2**	**2**

Table Two
Coded Configurations of The Transition Parameter Root, t
Including Illegal Combinations

	Denary	Ternary		
		9	3	1
t_1	0	0	0	0
t_2	3	0	1	0
t_3	6	0	2	0
t_4	10	1	0	1
t_5	13	1	1	1
t_6	16	1	2	1
t_7	20	2	0	2
t_8	23	2	1	2
t_9	26	2	2	2

Table Three
Coded Configuration of The Symmetric Examples of
The Transition Parameter Root, t

Notation

a	Generalised Radius Function #1
a_i	A Polygon Sector Area
A	The Exact Area
A_e	The Economical Polygonal Area
b	Generalised Radius Function #2
C_A	The Areal Corrector
C_P	The Perimetric Corrector
θ	The InterAngle
θ_i	a Particular InterAngle
k	The Estimation Constant
μ_r	The Mean Radius
μ_t	The Mean Radial Length Transition
m	any Metric of Central Tendency
n	The Number of Polygon Sides or Vertices
π	The Ludolphine Constant
p	a Perimetric Element
p_i	a Polygon Side Length
P	The Exact Perimeter
P_e	The Economical Polygonal Perimeter
P_{est}	The Estimated Perimeter
r	The Generalised Radius
r_1	The First Radius
r_2	The Second Radius
r_c	The Characteristic Radius
r_i	a Particular Radius
σ	The Standard Deviation
s	any Metric of Dispersion
τ_t	The Sum of Radius Length Transitions
t	a Radius Length Transition
T	The Radial Length Transition Parameter
T_X	The Extended Transition Parameter