Economisation of the Computation of The Momental Statistics of Symmetric Matrices

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A SYMMETRIC MATRIX contains numbers which characterise the relation of paired entities A, B such that the characteristic is independent of entity order (i.e. A to B or B to A).

For example, d is a property of the association of A and B contained in matrix [M] and illustrated by:-

0	1	2	3
1	0	4	5
2	4	0	6
3	5	6	0

Table One

where A and i may be read horizontally and B and j vertically such that:-

$d_{ij}=f(A_i,B_j)$
$d_{1,2}=1$
$d_{2,3}=4$
$d_{3,3}=0$

For a general symmetric matrix, $m=m_A=m_B=n^{½}$ where n is The Population of Properties, d.

Because the numbers in the upper right of the matrix mirror those in the bottom left, the Mean and the Variances of *all* d's in the body of [M] can be computed by recourse to summations of the non-diagonal numbers in the upper right only.

<u>The Statistics for All n Elements</u>

By use of the usual formulae the descriptive statistics for the n d values of Table One are:-

$\mu = 2.625$
$V_\sigma = 4.484375$
$V_s = 4.783333'$
and these may be employed for testing and for reference.

The Summation through The Upper Right

Let k, the Number of Upper Right Elements, be:-

$$k = \frac{m^2 - m}{2} \qquad \textbf{Eqn.1}$$

The Sum of the Upper Right Elements, S_k, is given by:-

$$S_k = \sum_{j=1}^{m_B-1} \sum_{i=j+1}^{m_A} d_{ij}$$

$$= \sum_{j=1}^{m} \sum_{i=j+1}^{m} d_{ij} \qquad \textbf{Eqn.2}$$

from which the Mean of the Upper Right Elements, μ_k, may be defined as:-

$$\mu_k = \frac{S_k}{k} \qquad \textbf{Eqn.3}$$

For the example of *Table One only* we may for confirmatory purposes use:-

$$S_k = \frac{n(n+1)}{2} = \frac{6x7}{2} = 21 \qquad \textbf{Eqn.4}$$

and:-

$$\mu_k = \frac{S_k}{k} = \frac{21}{6} = 3.5 \qquad \textbf{Eqn.5}$$

Because the diagonal elements are zero by definition it follows that:-

$$\sum d = 2\, S_k \qquad \textbf{Eqn.6}$$

(which is 42 for Table One).

The Mean of All n Elements
By Use of S_k

To transform the economical μ_k to the whole-matrix Mean, μ_n, we need to employ:-

$$\mu_n = \frac{m-1}{m}\mu_k \qquad \textbf{Eqn.7}$$

giving μ_n as 2.625 for the example.

The Sum of Deviations

Let ΣD_k^2 be the Sum of the Squared Deviations for the k elements of the Upper Right only. Let ΣD_n^2 be the Sum of the Squared Deviations for the Entire Matrix. Further allow ΣD_0^2 to be the Sum of the Squared Deviations for the Diagonal Elements only. Then:-

$$\Sigma D_0^2 = \sum_{i=1}^{m}(0-\mu_n)^2 = m\mu_n^2 \qquad \textbf{Eqn.8}$$

It accordingly follows that:-

$$\begin{aligned}\Sigma D_n^2 &= 2\Sigma D_k^2 + \Sigma D_0^2 \\ &= 2\sum_{j=1}^{m}\sum_{i=j+1}^{m}(d_{ij}-\mu_n)^2 + m\mu_n^2\end{aligned} \qquad \textbf{Eqn.9}$$

For Table One ΣD_0^2 is 27.5625 whilst ΣD_k^2 is 22.09375. Hence ΣD_n^2=71.75.

The Variances

The Population Variance, V_σ, is given by:-

$$V_\sigma = \frac{\Sigma D_n^2}{n} \qquad \textbf{Eqn.10}$$

or 71.75/16 = 4.484375 for Table One.
Sample Variance, V_s, is given by:-

$$V_s = \frac{n}{n-1} x \frac{\Sigma D_n^2}{n} = \frac{\Sigma D_n^2}{n-1} \qquad \textbf{Eqn.11}$$

This is 71.75/15 = 4.783333' for Table One.

Notation

A	The First Characterised Entity
B	The Second Characterised Entity
d_{ij}	a Property of Association between A and B
i	The Serial Number of Entities A
j	The Serial Number of Entities B
k	The Number of Upper Right Elements
μ	The Arithmetic Mean
μ_k	The Arithmetic Mean of Upper Right Elements
μ_n	The Whole-Population Arithmetic Mean
m	The Matrix Side Length
m_A	The Number of Entities A
m_B	The Number of Entities B
n	The Number of All Matrix Elements
Σd	The Sum of Non-Diagonal Elements
ΣD_0^2	The Sum of Diagonal Squared Deviations
ΣD_k^2	The Sum of Upper Right Squared Deviations
ΣD_n^2	The Sum of All Squared Deviations
S_k	The Sum of Upper Right Elements
V_σ	The Population Variance
V_s	The Sample Variance