Corroborative Approximation of
The Mean Transect for Many-Sided Regular Polygons

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When a regular polygon has a large number of sides the ratio $t/A^{0.5}$ should converge to a limit representative of that appertaining to a circle.

If we compute polygonal statistics for *the unit side* then the Perpendicular Altitude, a, of the isosceles triangle based upon the unit side is given by:-

$$a = r\,\mathrm{Cos}\frac{\pi}{\mathrm{n}} \qquad \textbf{Eqn.1}$$

Now as The Number of Sides, n, tends toward infinity:-

$$\mathrm{Cos}\frac{\pi}{\mathrm{n}} \approx 1 - \frac{\pi^2}{2\,\mathrm{n}^2} \qquad \textbf{Eqn.2}$$

and:-

$$a \approx r \qquad \textbf{Eqn.3}$$

Meanwhile the exact area of the triangle based upon the unit side is given by:-

$$\alpha = \frac{1xa}{2} = \frac{a}{2} \qquad \textbf{Eqn.4}$$

Accordingly the exact Area of a Regular Polygon, A, is given by:-

$$A = n\alpha = \frac{na}{2} = n\left[\frac{r}{2}\mathrm{Cos}\frac{\pi}{\mathrm{n}}\right] \qquad \textbf{Eqn.5}$$

Further the Area of a Circle of Radius r that is A_c is given by:-

$$A_c = \pi\, r^2 \qquad \textbf{Eqn.6}$$

For high n we may approximate r by equivalating Equations Six and Five as:-

$$\pi r^2 \approx n\left[\frac{r}{2}\mathrm{Cos}\frac{\pi}{\mathrm{n}}\right] \qquad \textbf{Eqn.7}$$

Substituting Equation Two:-

$$\pi r^2 \approx n\left[\frac{r}{2}\left(1 - \frac{\pi^2}{2\,n^2}\right)\right] \qquad \textbf{Eqn.8}$$

$$\therefore \pi r^2 \approx \frac{nr}{2} - \frac{nr\pi^2}{4n^2} \qquad \textbf{Eqn.9}$$

Dividing all terms by πr^2 gives:-

$$1 \approx \frac{n}{2\pi r} - \frac{\pi}{4nr} \qquad \textbf{Eqn.10}$$

$$\therefore 1 \approx \frac{1}{2r}\left(\frac{n}{\pi} - \frac{\pi}{2n}\right) \qquad \textbf{Eqn.11}$$

$$\therefore r \approx \frac{1}{2}\left(\frac{n}{\pi} - \frac{\pi}{2n}\right) \qquad \textbf{Eqn.12}$$

Since $a \approx r$ for high n we may write the Approximate Polygon Area, A_a, as:-

$$A_a \approx \frac{nr}{2} \qquad \textbf{Eqn.13}$$

And the Error Discrepancy, ε, as:-

$$\varepsilon = 100\left(\frac{A_a - A_c}{A_c}\right) \qquad \textbf{Eqn.14}$$

For n=25, ε=0.79585214879% dropping to ε=0.0493723863% at n=100.
Substitution of Equation Twelve in Equation Thirteen allows us to write:-

$$A_a \approx \frac{n^2}{4\pi} - \frac{\pi}{8} \qquad \textbf{Eqn.15}$$

Dividing all terms by π gives:-

$$\frac{A_a}{\pi} \approx \frac{n^2}{4\pi^2} - \frac{1}{8} \qquad \textbf{Eqn.16}$$

or:-

$$A_a \approx \pi\left(\frac{n^2}{4\pi^2}\right) \qquad \textbf{Eqn.17}$$

Accordingly:-

$$\sqrt{A_a} \approx \sqrt{\pi}\left(\frac{n}{2\pi}\right) \approx \frac{n}{2\sqrt{\pi}} \qquad \textbf{Eqn.18}$$

Error inherent in this approximation for n=100 is about 0.05% relative to the A_a of Equation Thirteen.

Let f(n) be a convergent function of n which yields an exact or approximate value for $t/A^{0.5}$ for any Number of Sides, n. As $n \rightarrow \infty$, $f(n) \rightarrow k$, where k is some constant.

For example, f(n) could be given by:-

$$f(n) = e^{u-\frac{v}{n}}$$ **Eqn.19**

where as $n \to \infty$, $k \to e^{a}$.
In such a context:-

$$t = \frac{n}{2\sqrt{\pi}} \cdot e^{u-\frac{v}{n}}$$ **Eqn.20**

And The Mean Transect of a Many-Sided Polygon, or indeed a Circle, may be approximated.

Notation

α	The Area of a Component Triangle
a	The Perpendicular Altitude of a Component Triangle
A	The (Exact) Area of a Regular Polygon
A_a	The (Approximate) Area of a Regular Polygon
A_c	The Area of a Circle of Radius r
b	a Decay Gradient
ε	The Percentage Relative Error
f(n)	a Function of n
k	The Limiting Value of f(n)
n	The Number of Polygon Sides
π	The Ludolphine Constant
r	The (Center to Vertex) Polygon Radius
u	a Decay Intercept
v	a Decay Gradient